Section 1.5 Inverse Functions

- Two functions f and g are inverses of each other if $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f.
- Be able to find the inverse of a function, if it exists.
  1. Replace $f(x)$ with y.
  2. Interchange x and y.
  3. Solve for y. If this equation represents y as a function of x, then you have found $f^{-1}(x)$. If this equation does not represent y as a function of x, then f does not have an inverse function.
- A function f has an inverse function if and only if no **horizontal** line crosses the graph of f at more than one point.
- A function f has an inverse function if and only if f is one-to-one.

Solutions to Odd-Numbered Exercises

1. The inverse is a line through $(-1, 0)$.
Matches graph (c).

3. The inverse is half a parabola starting at $(1, 0)$.
Matches graph (a).

5. $f^{-1}(x) = \dfrac{x}{8} = \dfrac{1}{8}x$

$f(f^{-1}(x)) = f\left(\dfrac{x}{8}\right) = 8\left(\dfrac{x}{8}\right) = x$

$f^{-1}(f(x)) = f^{-1}(8x) = \dfrac{8x}{8} = x$

7. $f^{-1}(x) = x - 10$

$f(f^{-1}(x)) = f(x - 10) = (x - 10) + 10 = x$

$f^{-1}(f(x)) = f^{-1}(x + 10) = (x + 10) - 10 = x$

9. $f^{-1}(x) = \dfrac{x - 1}{2}$

$f(f^{-1}(x)) = f\left(\dfrac{x-1}{2}\right) = 2\left(\dfrac{x-1}{2}\right) + 1 = (x - 1) + 1 = x$

$f^{-1}(f(x)) = f^{-1}(2x + 1) = \dfrac{(2x + 1) - 1}{2} = \dfrac{2x}{2} = x$

11. $f^{-1}(x) = x^3$

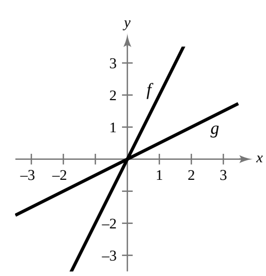
$f(f^{-1}(x)) = f(x^3) = \sqrt[3]{x^3} = x$

$f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$

13. (a) $f(g(x)) = f\left(\dfrac{x}{2}\right) = 2\left(\dfrac{x}{2}\right) = x$

$g(f(x)) = g(2x) = \dfrac{2x}{2} = x$

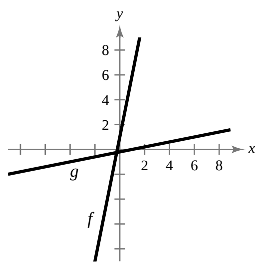
(b)



15. (a) $f(g(x)) = f\left(\dfrac{x-1}{5}\right) = 5\left(\dfrac{x-1}{5}\right) + 1 = x$

$g(f(x)) = g(5x+1) = \dfrac{(5x+1)-1}{5} = x$

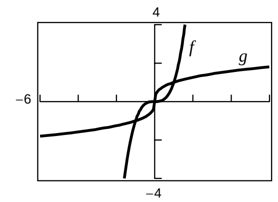
(b)



17. (a) $f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$

$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

(b)



Reflections in the line $y = x$

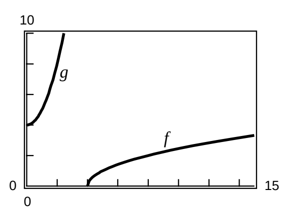
19. (a) $f(g(x)) = f(x^2+4),\ x \geq 0$

$= \sqrt{(x^2+4)-4} = x$

$g(f(x)) = g\left(\sqrt{x-4}\right)$

$= \left(\sqrt{x-4}\right)^2 + 4 = x$

(b)

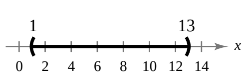


Reflections in the line $y = x$

21. (a) $f(g(x)) = f\left(\sqrt[3]{1-x}\right) = 1 - \left(\sqrt[3]{1-x}\right)^3 = 1 - (1-x) = x$

$g(f(x)) = g(1-x^3) = \sqrt[3]{1-(1-x^3)} = \sqrt[3]{x^3} = x$

(b)



Reflections in the line $y = x$

23. (a) $f(g(x)) = f\left(-\dfrac{2x+6}{7}\right) = -\dfrac{7}{2}\left(-\dfrac{2x+6}{7}\right) - 3 = \dfrac{2x+6}{2} - 3 = (x+3) - 3 = x$

$g(f(x)) = g\left(-\dfrac{7}{2}x - 3\right) = -\dfrac{2\left(-\frac{7}{2}x - 3\right) + 6}{7} = -\dfrac{-7x - 6 + 6}{7} = \dfrac{7x}{7} = x$

(b)

x	2	0	−2	−4	−6
$f(x)$	−10	−3	4	11	18

x	−10	−3	4	11	18
$g(x)$	2	0	−2	−4	−6

Note that the entries in the tables are the same except that the rows are interchanged.

25. (a) $f(g(x)) = f\left(\sqrt[3]{x-5}\right) = \left[\sqrt[3]{x-5}\right]^3 + 5 = (x-5) + 5 = x$

$g(f(x)) = g(x^3 + 5) = \sqrt[3]{(x^3+5) - 5} = \sqrt[3]{x^3} = x$

(b)

x	-3	-2	-1	0	1
$f(x)$	-22	-3	4	5	6

x	-22	-3	4	5	6
$g(x)$	-3	-2	-1	0	1

Note that the entries in the tables are the same except that the rows are interchanged.

27. (a) $f(g(x)) = f(8 + x^2) = -\sqrt{(8+x^2) - 8} = -\sqrt{x^2} = -(-x) = x$

$\left[\text{Since } x \le 0,\ \sqrt{x^2} = -x\right]$

$g(f(x)) = g\left(-\sqrt{x-8}\right) = 8 + \left[-\sqrt{x-8}\right]^2 = 8 + (x-8) = x$

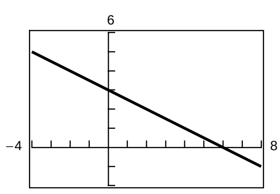
(b)

x	8	9	12	17	24
$f(x)$	0	-1	-2	-3	-4

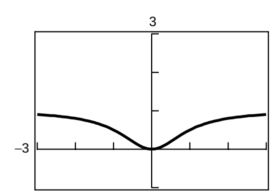
x	0	-1	-2	-3	-4
$g(x)$	8	9	12	17	24

Note that the entries in the tables are the same except that the rows are interchanged.

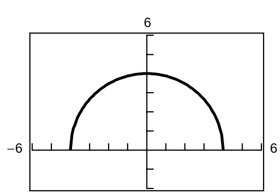
29. Since no horizontal line crosses the graph of f at more than one point, f **has** an inverse.



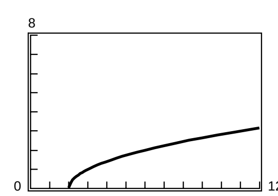
31. Since some horizontal lines cross the graph of f twice, f does **not** have an inverse.



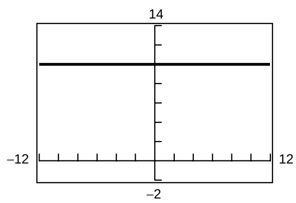
33. No, because some horizontal lines intersect the graph more than once, h does not have an inverse.



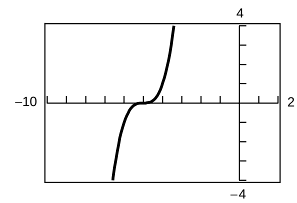
35. Yes, because no horizontal lines intersect the graph at more than one point, f has an inverse.



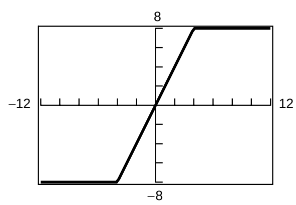
37. f does not pass the horizontal line test, so f has no inverse.



39. g passes the horizontal line test, so g has an inverse.



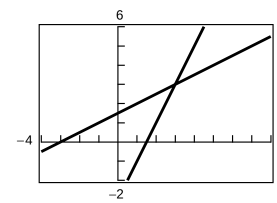
41. $h(x) = |x + 4| - |x - 4|$



h does not pass the horizontal line test, so h does not have an inverse.

43.

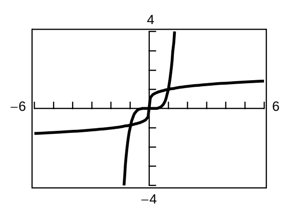
$$f(x) = 2x - 3$$
$$y = 2x - 3$$
$$x = 2y - 3$$
$$y = \frac{x + 3}{2}$$
$$f^{-1}(x) = \frac{x + 3}{2}$$



Reflections in the line $y = x$

45.

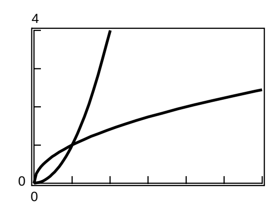
$$f(x) = x^5$$
$$y = x^5$$
$$x = y^5$$
$$y = \sqrt[5]{x}$$
$$f^{-1}(x) = \sqrt[5]{x}$$



Reflections in the line $y = x$

47.

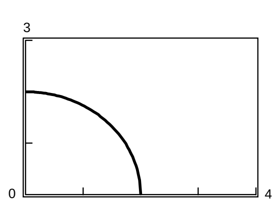
$$f(x) = \sqrt{x}$$
$$y = \sqrt{x}$$
$$x = \sqrt{y}$$
$$y = x^2$$
$$f^{-1}(x) = x^2, \; x \geq 0$$



Reflections in the line $y = x$

49.

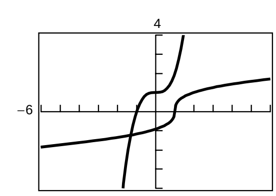
$$f(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2$$
$$y = \sqrt{4 - x^2}$$
$$x = \sqrt{4 - y^2}$$
$$x^2 = 4 - y^2$$
$$y^2 = 4 - x^2$$
$$y = \sqrt{4 - x^2}$$
$$f^{-1}(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2$$



Reflections in the line $y = x$

51.

$$f(x) = \sqrt[3]{x - 1}$$
$$y = \sqrt[3]{x - 1}$$
$$x = \sqrt[3]{y - 1}$$
$$x^3 = y - 1$$
$$y = x^3 + 1$$
$$f^{-1}(x) = x^3 + 1$$



Reflections in the line $y = x$

53. $f(x) = \dfrac{4}{x}$

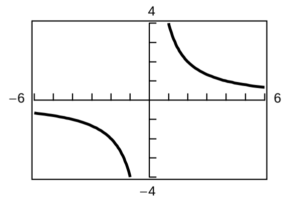
$y = \dfrac{4}{x}$

$x = \dfrac{4}{y}$

$xy = 4$

$y = \dfrac{4}{x}$

$f^{-1}(x) = \dfrac{4}{x}$



Reflections in the line $y = x$

55. $f(x) = x^4$

$y = x^4$

$x = y^4$

$y = \pm\sqrt[4]{x}$

f is not one-to-one.

This does not represent y as a function of x. f does not have an inverse.

57. $f(x) = \dfrac{3x + 4}{5}$

$y = \dfrac{3x + 4}{5}$

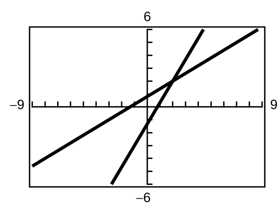
$x = \dfrac{3y + 4}{5}$

$5x = 3y + 4$

$5x - 4 = 3y$

$(5x - 4)/3 = y$

$f^{-1}(x) = \dfrac{5x - 4}{3}$



f is one-to-one and has an inverse.

59. $f(x) = (x + 3)^2,\ x \geq -3,\ y \geq 0$

$y = (x + 3)^2,\ x \geq -3,\ y \geq 0$

$x = (y + 3)^2,\ y \geq -3,\ x \geq 0$

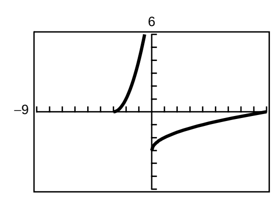
$\sqrt{x} = y + 3\quad,\ y \geq -3,\ x \geq 0$

$y = \sqrt{x} - 3,\ x \geq 0,\ y \geq -3$

f is one-to-one.

This is a function of x, so f has an inverse.

$f^{-1}(x) = \sqrt{x} - 3,\ x \geq 0$



61. $h(x) = \dfrac{4}{x^2}$ is not one-to-one, and does not have an inverse. For example, $h(1) = h(-1) = 4$.

63. $f(x) = \sqrt{2x + 3} \implies x \geq -\dfrac{3}{2},\ y \geq 0$

$y = \sqrt{2x + 3},\ x \geq -\dfrac{3}{2},\ y \geq 0$

$x = \sqrt{2y + 3},\ y \geq -\dfrac{3}{2},\ x \geq 0$

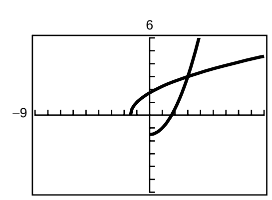
$x^2 = 2y + 3,\ x \geq 0,\ y \geq -\dfrac{3}{2}$

$y = \dfrac{x^2 - 3}{2},\ x \geq 0,\ y \geq -\dfrac{3}{2}$

f is one to one.

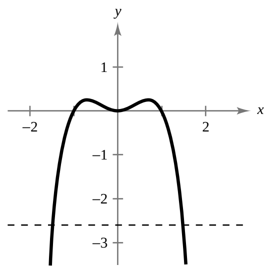
This is a function of x, so f has an inverse.

$f^{-1}(x) = \dfrac{x^2 - 3}{2},\ x \geq 0$



65. $g(x) = x^2 - x^4$

The graph fails the horizontal line test, so g does not have an inverse. g is not one-to-one.



67. $f(x) = ax + b, a \neq 0$

$$y = ax + b$$
$$x = ay + b$$
$$x - b = ay$$
$$y = (x - b)/a$$

f is one-to-one and has an inverse, $f^{-1}(x) = \dfrac{x - b}{a}$.

69. If we let $f(x) = (x - 2)^2$, $x \geq 2$, then f has an inverse. [Note: we could also let $x \leq 2$.]

$$f(x) = (x - 2)^2, \; x \geq 2, \; y \geq 0$$
$$y = (x - 2)^2, \; x \geq 2, \; y \geq 0$$
$$x = (y - 2)^2, \; x \geq 0, \; y \geq 2$$
$$\sqrt{x} = y - 2, \quad x \geq 0, \; y \geq 2$$
$$\sqrt{x} + 2 = y, \quad x \geq 0, \; y \geq 2$$

Thus, $f^{-1}(x) = \sqrt{x} + 2, \; x \geq 0$.

71. If we let $f(x) = |x + 2|$, $x \geq -2$, then f has an inverse. [Note: we could also let $x \leq -2$.]

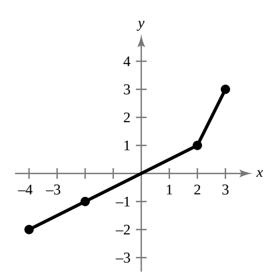
$$f(x) = |x + 2|, \; x \geq -2$$
$$f(x) = x + 2 \text{ when } x \geq -2$$
$$y = x + 2, \; x \geq -2, \; y \geq 0$$
$$x = y + 2, \; x \geq 0, \; y \geq -2$$
$$x - 2 = y, \quad x \geq 0, \; y \geq -2$$

Thus, $f^{-1}(x) = x - 2, \; x \geq 0$.

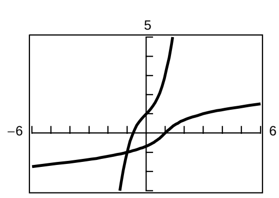
73.

x	$f(x)$
-2	-4
-1	-2
1	2
3	3

x	$f^{-1}(x)$
-4	-2
-2	-1
2	1
3	3

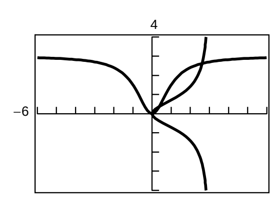


75. $f(x) = x^3 + x + 1$



The graph of the inverse relation is an inverse function since it satisfies the vertical line test.

77. $g(x) = \dfrac{3x^2}{x^2 + 1}$



The graph of the inverse relation is not an inverse function since it does not satisfy the vertical line test.

In Exercises 79, 81, and 83, $f(x) = \frac{1}{8}x - 3$, $f^{-1}(x) = 8(x + 3)$, $g(x) = x^3$, $g^{-1}(x) = \sqrt[3]{x}$.

79. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}\left(\sqrt[3]{1}\right) = 8\left(\sqrt[3]{1} + 3\right) = 8(1 + 3) = 32$

81. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(8[6 + 3]) = f^{-1}(72) = 8(72 + 3) = 600$

83. $(f \circ g)(x) = f(g(x)) = f(x^3) = \frac{1}{8}x^3 - 3$. Now find the inverse of $(f \circ g)(x) = \frac{1}{8}x^3 - 3$:

$$y = \tfrac{1}{8}x^3 - 3$$
$$x = \tfrac{1}{8}y^3 - 3$$
$$x + 3 = \tfrac{1}{8}y^3$$
$$8(x + 3) = y^3$$
$$\sqrt[3]{8(x + 3)} = y$$
$$(f \circ g)^{-1}(x) = 2\sqrt[3]{x + 3}$$

Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

In Exercises 85 and 87, $f(x) = x + 4,\ f^{-1}(x) = x - 4,\ g(x) = 2x - 5,\ g^{-1}(x) = \dfrac{x + 5}{2}$.

85. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) = g^{-1}(x - 4) = \dfrac{(x - 4) + 5}{2} = \dfrac{x + 1}{2}$

87. $(f \circ g)(x) = f(g(x)) = f(2x - 5) = (2x - 5) + 4 = 2x - 1$ Now find the inverse of $(f \circ g)(x) = 2x - 1$:

$$y = 2x - 1$$
$$x = 2y - 1$$
$$x + 1 = 2y$$
$$y = \frac{x + 1}{2}$$
$$(f \circ g)^{-1}(x) = \frac{x + 1}{2}$$

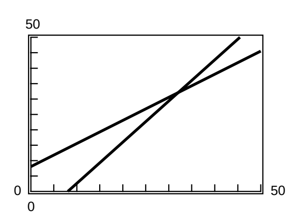
Note that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$; see Exercise 94.

89. (a)

$$y = 8 + 0.75x$$
$$x = 8 + 0.75y$$
$$x - 8 = 0.75y$$
$$\frac{x - 8}{0.75} = y$$
$$y = f^{-1}(x) = \frac{x - 8}{0.75}$$

$x =$ hourly wage
$y =$ number of units produced

(b)



(c) If 10 units are produced, then $y = 8 + 0.75(10) = \$15.50$.

(d) If the hourly wage is \$22.25, then

$$y = \frac{22.25 - 8}{0.75} = 19 \text{ units.}$$