CHAPTER 2
Polynomial and Rational Functions

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Section 2.1 Quadratic Functions

You should know the following facts about parabolas.

- $f(x) = ax^2 + bx + c, \ a \neq 0$, is a quadratic function, and its graph is a parabola.
- If $a > 0$, the parabola opens upward and the vertex is the minimum point. If $a < 0$, the parabola opens downward and the vertex is the maximum point.
- The vertex is $(-b/2a, f(-b/2a))$.
- To find the x-intercepts (if any), solve

  $$ax^2 + bx + c = 0.$$
- The standard form of the equation of a parabola is

  $$f(x) = a(x - h)^2 + k$$

  where $a \neq 0$.

  (a) The vertex is (h, k).

  (b) The axis is the vertical line $x = h$.

Solutions to Odd-Numbered Exercises

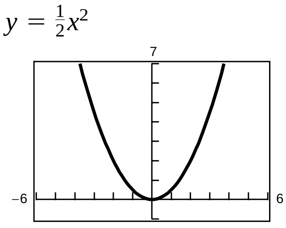
1. $f(x) = (x - 2)^2$ opens upward and has vertex $(2, 0)$. Matches graph (g).

3. $f(x) = x^2 - 2$ opens upward and has vertex $(0, -2)$. Matches graph (b).

5. $f(x) = 4 - (x - 2)^2 = -(x - 2)^2 + 4$ opens downward and has vertex $(2, 4)$. Matches graph (f).

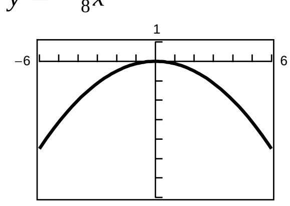
7. $f(x) = x^2 + 3$ opens upward and has vertex $(0, 3)$. Matches graph (e).

9. (a) $y = \frac{1}{2}x^2$



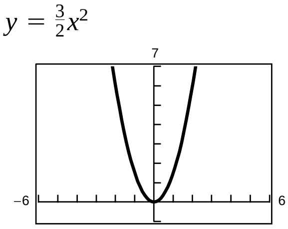
Vertical shrink

(b) $y = -\frac{1}{8}x^2$



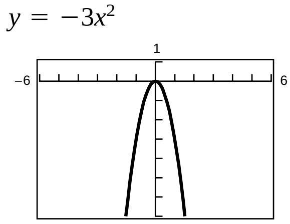
Vertical shrink and reflection in the x-axis

(c) $y = \frac{3}{2}x^2$



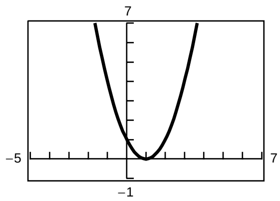
Vertical stretch

(d) $y = -3x^2$



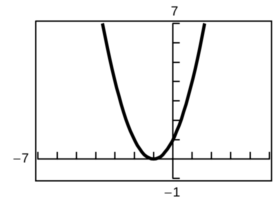
Vertical stretch and reflection in the x-axis

11. (a) $y = (x - 1)^2$



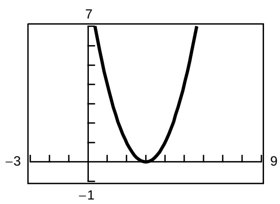
Horizontal shift one unit to the right

(b) $y = (x + 1)^2$



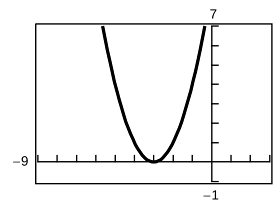
Horizontal shift one unit to the left.

(c) $y = (x - 3)^2$



Horizontal shift three units to the right

(d) $y = (x + 3)^2$

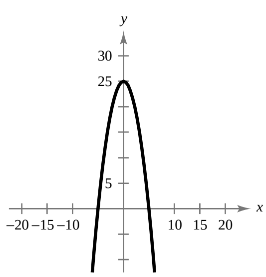


Horizontal shift three units to the left

13. $f(x) = 25 - x^2$

Vertex: $(0, 25)$

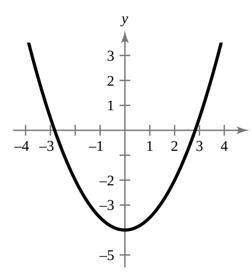
Intercepts: $(-5, 0)$, $(0, 25)$, $(5, 0)$



15. $f(x) = \frac{1}{2}x^2 - 4$

Vertex: $(0, -4)$

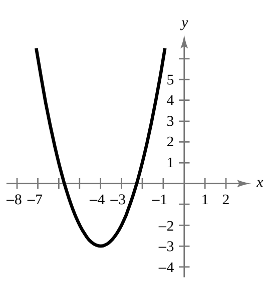
Intercepts: $\left(\pm 2\sqrt{2}, 0\right)$, $(0, -4)$



17. $f(x) = (x + 4)^2 - 3$

Vertex: $(-4, -3)$

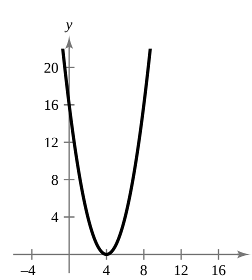
Intercepts: $(0, 13)$, $(-4 \pm \sqrt{3}, 0)$



19. $h(x) = x^2 - 8x + 16 = (x - 4)^2$

Vertex: $(4, 0)$

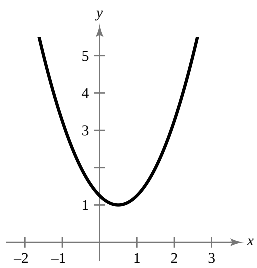
Intercepts: $(0, 16)$, $(4, 0)$



21. $f(x) = x^2 - x + \frac{5}{4} = \left(x - \frac{1}{2}\right)^2 + 1$

Vertex: $\left(\frac{1}{2}, 1\right)$

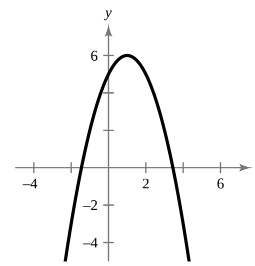
Intercepts: $\left(0, \frac{5}{4}\right)$



23. $f(x) = -x^2 + 2x + 5 = -(x - 1)^2 + 6$

Vertex: $(1, 6)$

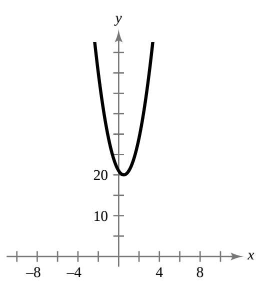
Intercepts: $\left(1 - \sqrt{6}, 0\right)$, $(0, 5)$, $\left(1 + \sqrt{6}, 0\right)$



25. $h(x) = 4x^2 - 4x + 21 = 4\left(x - \frac{1}{2}\right)^2 + 20$

Vertex: $\left(\frac{1}{2}, 20\right)$

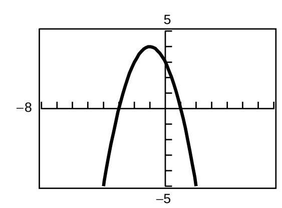
Intercept: $(0, 21)$



27. $f(x) = -(x^2 + 2x - 3) = -(x + 1)^2 + 4$

Vertex: $(-1, 4)$

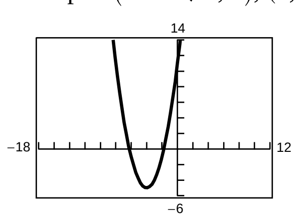
Intercepts: $(-3, 0)$, $(0, 3)$, $(1, 0)$



29. $g(x) = x^2 + 8x + 11 = (x + 4)^2 - 5$

Vertex: $(-4, -5)$

Intercepts: $\left(-4 \pm \sqrt{5}, 0\right)$, $(0, 11)$

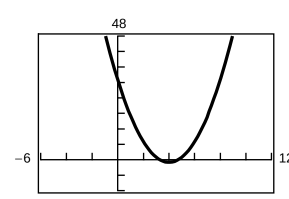


31. $f(x) = 2x^2 - 16x + 31$

$= 2(x - 4)^2 - 1$

Vertex: $(4, -1)$

Intercepts: $\left(4 \pm \frac{1}{2}\sqrt{2}, 0\right)$, $(0, 31)$

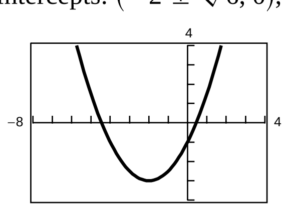


33. $g(x) = \frac{1}{2}(x^2 + 4x - 2) = \frac{1}{2}(x^2 + 4x + 4 - 6)$

$= \frac{1}{2}(x + 2)^2 - 3$

Vertex: $(-2, -3)$

Intercepts: $\left(-2 \pm \sqrt{6}, 0\right)$, $(0, -1)$



35. $(1, 0)$ is the vertex.

$f(x) = a(x - 1)^2 + 0 = a(x - 1)^2$

Since the graph passes through the point $(0, 1)$ we have:

$1 = a(0 - 1)^2$

$1 = a$

$f(x) = 1(x - 1)^2 = (x - 1)^2$

37. $(-1, 4)$ is the vertex.

$f(x) = a(x + 1)^2 + 4$

Since the graph passes through the point $(1, 0)$ we have

$$0 = a(1 + 1)^2 + 4$$
$$0 = 4a + 4$$
$$-1 = a$$

Thus, $f(x) = -(x + 1)^2 + 4$. Note that $(-3, 0)$ is on the parabola.

39. $(-2, 5)$ is the vertex.

$f(x) = a(x + 2)^2 + 5$

Since the graph passes through the point $(0, 9)$, we have:

$$9 = a(0 + 2)^2 + 5$$
$$4 = 4a$$
$$1 = a$$

$f(x) = 1(x + 2)^2 + 5 = (x + 2)^2 + 5$

41. $(3, 4)$ is the vertex.

$f(x) = a(x - 3)^2 + 4$

Since the graph passes through the point $(1, 2)$, we have:

$$2 = a(1 - 3)^2 + 4$$
$$-2 = 4a$$
$$-\tfrac{1}{2} = a$$
$$f(x) = -\tfrac{1}{2}(x - 3)^2 + 4$$

43. $(-2, -2)$ is the vertex

$f(x) = a(x + 2)^2 - 2$

Since the graph passes through $(-1, 0)$,

$$0 = a(-1 + 2)^2 - 2$$
$$0 = a - 2$$
$$2 = a$$

Thus, $f(x) = 2(x + 2)^2 - 2$

45. $\left(\frac{5}{2}, -\frac{3}{4}\right)$ is the vertex.

$f(x) = a\left(x - \frac{5}{2}\right)^2 - \frac{3}{4}$

Since the graph passes through $(-2, 4)$,

$$4 = a\left(-2 - \tfrac{5}{2}\right)^2 - \tfrac{3}{4}$$
$$\tfrac{19}{4} = a\left(-\tfrac{9}{2}\right)^2$$
$$19 = 81a$$
$$a = \tfrac{19}{81}$$

Thus, $f(x) = \frac{19}{81}\left(x - \frac{5}{2}\right)^2 - \frac{3}{4}$

47. $y = x^2 - 16$

x-intercepts: $(\pm 4, 0)$

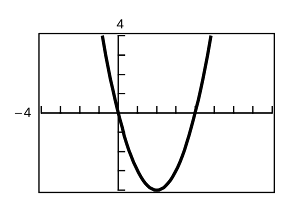
$$0 = x^2 - 16$$
$$x^2 = 16$$
$$x = \pm 4$$

49. $y = x^2 - 4x - 5$

x-intercepts: $(5, 0), (-1, 0)$

$$0 = x^2 - 4x - 5$$
$$0 = (x - 5)(x + 1)$$
$$x = 5 \text{ or } x = -1$$

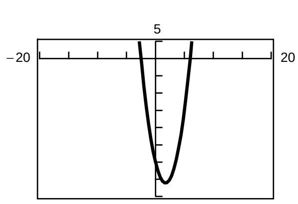
51. $y = x^2 - 4x$



x-intercepts: $(0, 0)$, $(4, 0)$

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$
$$x = 0 \text{ or } x = 4$$

53. $y = 2x^2 - 7x - 30$

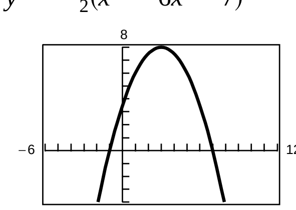


$$0 = 2x^2 - 7x - 30$$
$$0 = (2x + 5)(x - 6)$$
$$x = -\tfrac{5}{2} \text{ or } x = 6$$

x-intercepts:

$\left(-\frac{5}{2}, 0\right)$, $(6, 0)$

55. $y = -\frac{1}{2}(x^2 - 6x - 7)$



x-intercepts:

$(-1, 0), (7, 0)$

$$0 = -\tfrac{1}{2}(x^2 - 6x - 7)$$
$$0 = x^2 - 6x - 7$$
$$0 = (x + 1)(x - 7)$$
$$x = -1, 7$$

57. $f(x) = [x - (-1)](x - 3)$ opens upward
$= (x + 1)(x - 3)$
$= x^2 - 2x - 3$

$g(x) = -[x - (-1)](x - 3)$ opens downward
$= -(x + 1)(x - 3)$
$= -(x^2 - 2x - 3)$
$= -x^2 + 2x + 3$

Note: $f(x) = a(x + 1)(x - 3)$ has x-intercepts $(-1, 0)$ and $(3, 0)$ for all real numbers $a \neq 0$.

59. $f(x) = [x - (-3)]\left[x - \left(-\frac{1}{2}\right)\right](2)$ opens upward
$= (x + 3)\left(x + \frac{1}{2}\right)(2)$
$= (x + 3)(2x + 1)$
$= 2x^2 + 7x + 3$

$g(x) = -(2x^2 + 7x + 3)$ opens downward
$= -2x^2 - 7x - 3$

Note: $f(x) = a(x + 3)(2x + 1)$ has x-intercepts $(-3, 0)$ and $(-\frac{1}{2}, 0)$ for all real numbers $a \neq 0$.

61. Let $x =$ the first number and $y =$ the second number. Then the sum is

$$x + y = 110 \implies y = 110 - x.$$

The product is $P(x) = xy = x(110 - x) = 110x - x^2$.

$$\begin{aligned} P(x) &= -x^2 + 110x \\ &= -(x^2 - 110x + 3025 - 3025) \\ &= -[(x - 55)^2 - 3025] \\ &= -(x - 55)^2 + 3025 \end{aligned}$$

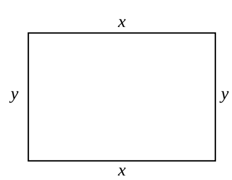
The maximum value of the product occurs at the vertex of $P(x)$ and is 3025. This happens when $x = y = 55$.

63. Let x be the first number and y be the second number.
Then $x + 2y = 24 \implies x = 24 - 2y$.
The product is $P = xy = (24 - 2y)y = 24y - 2y^2$.
Completing the square,

$$\begin{aligned} P &= -2y^2 + 24y \\ &= -2(y^2 - 12y + 36) + 72 \\ &= -2(y - 6)^2 + 72. \end{aligned}$$

The maximum value of the product P occurs at the vertex of the parabola and equals 72. This happens when $y = 6$ and $x = 24 - 2(6) = 12$.

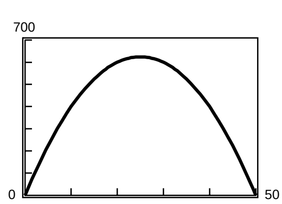
65.



$$2x + 2y = 100$$
$$y = 50 - x$$

(a) $A(x) = xy = x(50 - x)$
Domain: $0 < x < 50$

(b)

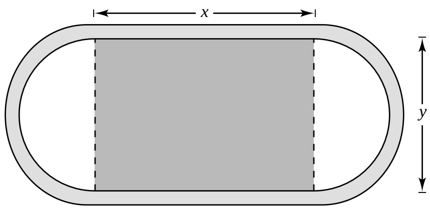


(c) The area is maximum (625 square feet) when $x = y = 25$. The rectangle has dimensions 25 ft $\times$ 25 ft. Algebraically, you have:

$$\begin{aligned} A(x) &= -(x^2 - 50x) \\ &= -(x^2 - 50x + 625) + 625 \\ &= -(x - 25)^2 + 625 \end{aligned}$$

$A(x)$ is a maximum of 625 when $x = 25$.

67. (a)



(b) Radius of semicircular ends of track: $r = \frac{1}{2}y$
distance around two semicircular parts of track:

$$d = 2\pi r = 2\pi\left(\frac{1}{2}y\right) = \pi y$$

(c) Distance traveled around track in one lap:

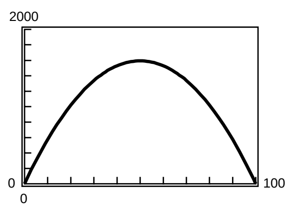
$$d = \pi y + 2x = 200$$
$$\pi y = 200 - 2x$$
$$y = \frac{200 - 2x}{\pi}$$

(d) Area of rectangular region:

$$A = xy = x\left(\frac{200 - 2x}{\pi}\right)$$
$$= \frac{1}{\pi}(200x - 2x^2)$$
$$= -\frac{2}{\pi}(x^2 - 100x)$$
$$= -\frac{2}{\pi}(x^2 - 100x + 2500 - 2500)$$
$$= -\frac{2}{\pi}(x - 50)^2 + \frac{5000}{\pi}$$

The area is maximum when $x = 50$ and $y = \frac{200 - 2(50)}{\pi} = \frac{100}{\pi}$.

(e)



The area is maximum when $x = 50$ and $y = \frac{200 - 2(50)}{\pi} = \frac{100}{\pi}$.

69. $C = 800 - 10x + 0.25x^2$

The minimum cost occurs at the vertex.

$$x = -\frac{b}{2a} = -\frac{(-10)}{2(0.25)} = \frac{10}{.5} = 20$$

$C(20) = 700$ is the minimum cost.

Graphically, you could graph $C = 800 - 10x + 0.25x^2$ in the window $[0, 40] \times [0, 1000]$ and find the vertex $(20, 700)$.

71. $P = -0.0002x^2 + 140x - 250{,}000$

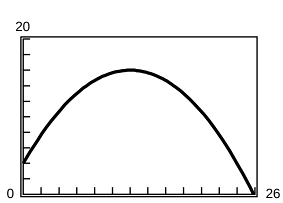
The vertex of this parabola is at

$$x = -\frac{b}{2a} = -\frac{140}{2(-0.0002)} = \frac{140}{0.0004}$$
$$= 350{,}000 \text{ units}$$

Thus, the maximum profit is attained at a sales level of 350,000 units.

73. $y = -\frac{1}{12}x^2 + 2x + 4$

(a)



(b) When $x = 0$, $y = 4$ feet.

(c) The vertex occurs at

$$x = -\frac{b}{2a} = -\frac{2}{2(-1/12)} = 12.$$

The maximum height is

$$y = -\frac{1}{12}(12)^2 + 2(12) + 4$$
$$= 16 \text{ feet.}$$

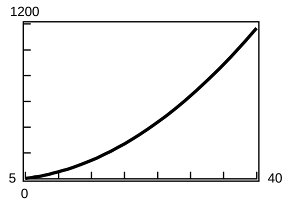
(d) You can solve this part graphically by finding the x-intercept of the graph:
$x \approx 25.856$.
Algebraically,

$$0 = -\frac{1}{12}x^2 + 2x + 4$$
$$0 = x^2 - 24x - 48 \quad \text{(Multiply both sides by } -12.)$$
$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(1)(-48)}}{2(1)}$$
$$= \frac{24 \pm \sqrt{768}}{2} = \frac{24 \pm 16\sqrt{3}}{2} = 12 \pm 8\sqrt{3}$$

Using the positive value for x, we have $x = 12 + 8\sqrt{3} \approx 25.86$ feet.

75. $V = 0.77x^2 - 1.32x - 9.31,\ 5 \le x \le 40$

(a)



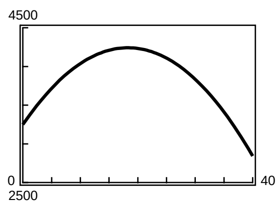
(b) $V(16) = 166.69$ board feet

(c) $500 = 0.77x^2 - 1.32x - 9.31$

$0 = 0.77x^2 - 1.32x - 509.31$

Using the Quadratic Formula and selecting the positive value for x, we have $x \approx 26.6$ inches in diameter. Or, use a graphing utility.

77. (a)



(b) Using a graphing utility, the maximum is approximately 4242 cigarettes at $t = 18.3$, or 1968. Yes, the warnings on cigarette packages seemed to have an effect.

(c) For 1960, $C(10) \approx 4038$ cigarettes per person. The annual consumption per smoker was

$$\frac{4038(116{,}530{,}000)}{48{,}500{,}000} = 9702 \text{ per smoker per year.}$$

The daily consumption per smoker was

$$\frac{9702}{365} \approx 26.6 \text{ cigarettes per smoker per day.}$$

79. True

$$-12x^2 - 1 = 0$$

$$12x^2 = -1 \quad \text{impossible}$$

81. Model (a) is preferable. $a > 0$ means the parabola opens upward and profits are increasing for t to the right of the vertex,

$$t \ge -\frac{b}{(2a)}.$$

83. $y = 3x - 10 = \frac{1}{4}x + 1$

$$12x - 40 = x + 4$$

$$11x = 44$$

$$x = 4$$

The graphs intersect at $(4, 2)$.

85. $y = x^3 + 2x - 1 = -2x + 15$

$$x^3 + 4x - 16 = 0$$

$$(x - 2)(x^2 + 2x + 8) = 0$$

$$x = 2$$

The graphs intersect at $(2, 11)$.

87. $y^2 = x^2 - 9$

$$y = \pm\sqrt{x^2 - 9}$$

No, y is not a function of x.

89. $x^2 + y^2 - 6x + 8y = 0$

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 25 \qquad \text{Circle}$$

y is not a function of x.