Section 2.2 Polynomial Functions of Higher Degree

- You should know the following basic principles about polynomials.
- $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ is a polynomial function of degree n.
- If f is of odd degree and
  - (a) $a_n > 0$, then
    1. $f(x) \to \infty$ as $x \to \infty$.
    2. $f(x) \to -\infty$ as $x \to -\infty$.
  - (b) $a_n < 0$, then
    1. $f(x) \to -\infty$ as $x \to \infty$.
    2. $f(x) \to \infty$ as $x \to -\infty$.
- If f is of even degree and
  - (a) $a_n > 0$, then
    1. $f(x) \to \infty$ as $x \to \infty$.
    2. $f(x) \to \infty$ as $x \to -\infty$.
  - (b) $a_n < 0$, then
    1. $f(x) \to -\infty$ as $x \to \infty$.
    2. $f(x) \to -\infty$ as $x \to -\infty$.
- The following are equivalent for a polynomial function.
  - (a) x = a is a zero of a function.
  - (b) x = a is a solution of the polynomial equation $f(x) = 0$.
  - (c) $(x - a)$ is a factor of the polynomial.
  - (d) $(a, 0)$ is an x-intercept of the graph of f.
- A polynomial of degree n has at most n distinct zeros.
- If f is a polynomial function such that a < b and $f(a) \neq f(b)$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.
- If you can find a value where a polynomial is positive and another value where it is negative, then there is at least one real zero between the values.

Solutions to Odd-Numbered Exercises

1. $f(x) = -2x + 3$ is a line with y-intercept (0, 3). Matches graph (f).

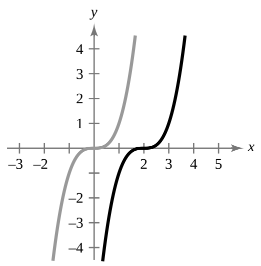
3. $f(x) = -2x^2 - 5x$ is a parabola with x-intercepts (0, 0) and $\left(-\frac{5}{2}, 0\right)$ and opens downward. Matches graph (c).

5. $f(x) = -\frac{1}{4}x^4 + 3x^2$ has intercepts (0, 0) and $\left(\pm 2\sqrt{3}, 0\right)$. Matches graph (e).

7. $f(x) = x^4 + 2x^3$ has intercepts (0, 0) and $(-2, 0)$. Matches graph (g).

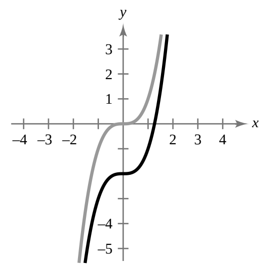
9. $y = x^3$

(a) $f(x) = (x - 2)^3$



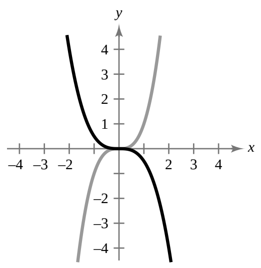
Horizontal shift two units to the right

(b) $f(x) = x^3 - 2$



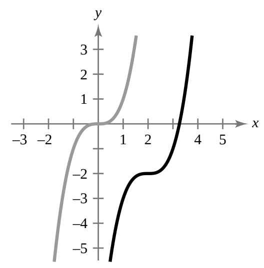
Vertical shift two units downward

(c) $f(x) = -\frac{1}{2}x^3$



Reflection in the x-axis and a vertical shrink

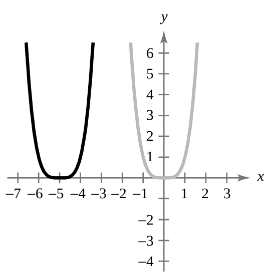
(d) $f(x) = (x - 2)^3 - 2$



Horizontal shift two units to the right and a vertical shift two units downward

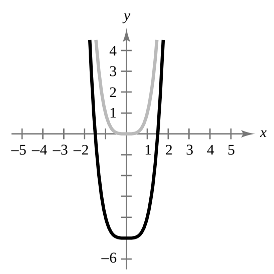
11. $y = x^4$

(a) $f(x) = (x + 5)^4$



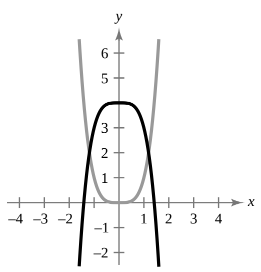
Horizontal shift five units to the left

(b) $f(x) = x^4 - 5$



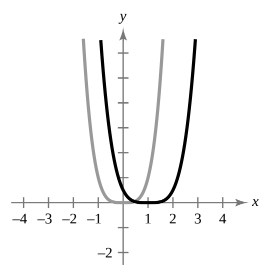
Vertical shift five units downward

(c) $f(x) = 4 - x^4$



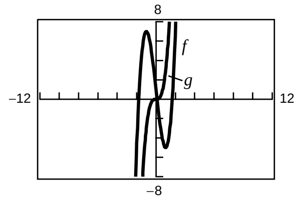
Reflection in the x-axis and then a vertical shift four units upward

(d) $f(x) = \frac{1}{2}(x - 1)^4$

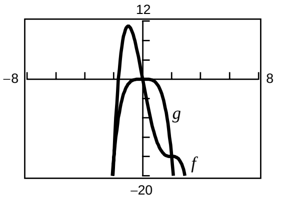


Horizontal shift one unit to the right and a vertical shrink

13. $f(x) = 3x^3 - 9x + 1;\ g(x) = 3x^3$



15. $f(x) = -(x^4 - 4x^3 + 16x);\ g(x) = -x^4$



17. $f(x) = 2x^2 - 3x + 1$

Degree: 2

Leading coefficient: 2

The degree is even and the leading coefficient is positive. The graph rises to the left and right.

19. $g(x) = 5 - \frac{7}{2}x - 3x^2$

Degree: 2

Leading coefficient: -3

The degree is even and the leading coefficient is negative. The graph falls to the left and right.

21. $f(x) = -2.1x^5 + 4x^3 - 2$

Degree: 5

Leading coefficient: -2.1

The degree is odd and the leading coefficient is negative. The graph rises to the left and falls to the right.

23. $f(x) = 6 - 2x + 4x^2 - 5x^3$

Degree: 3

Leading coefficient: -5

The degree is odd and the leading coefficient is negative. The graph rises to the left and falls to the right.

25. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

Degree: 2

Leading coefficient: $-\frac{2}{3}$

The degree is even and the leading coefficient is negative. The graph falls to the left and right.

27. $f(x) = x^2 - 25$

$= (x + 5)(x - 5)$

$x = \pm 5$

29. $h(t) = t^2 - 6t + 9$

$= (t - 3)^2$

$t = 3$

31. $f(x) = x^2 + x - 2$

$= (x + 2)(x - 1)$

$x = -2, 1$

33. $f(t) = t^3 - 4t^2 + 4t$

$= t(t - 2)^2$

$t = 0, 2$

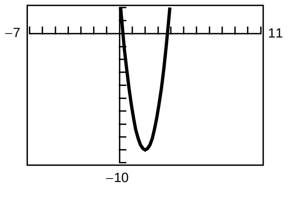
35. $f(x) = \dfrac{1}{2}x^2 + \dfrac{5}{2}x - \dfrac{3}{2}$

$= \dfrac{1}{2}(x^2 + 5x - 3)$

$$x = \frac{-5 \pm \sqrt{25 - 4(-3)}}{2} = -\frac{5}{2} \pm \frac{\sqrt{37}}{2}$$

$\approx 0.5414, -5.5414$

37. (a)



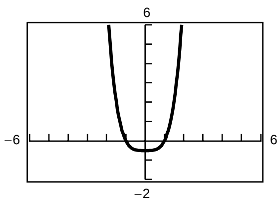
(b) $x \approx 3.732, 0.268$

(c) $f(x) = 3x^2 - 12x + 3$

$= 3(x^2 - 4x + 1)$

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

39. (a)



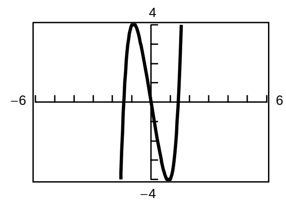
(b) $t = \pm 1$

(c) $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

$= \frac{1}{2}(t + 1)(t - 1)(t^2 + 1)$

$t = \pm 1$

41. (a)



(b) $x = 0, 1.414, -1.414$

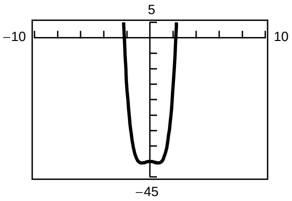
(c) $f(x) = x^5 + x^3 - 6x$

$= x(x^4 + x^2 - 6)$

$= x(x^2 + 3)(x^2 - 2)$

$x = 0, \pm\sqrt{2}$

43. (a)



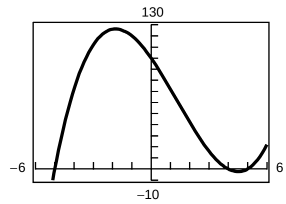
(b) $2.236, -2.236$

(c) $f(x) = 2x^4 - 2x^2 - 40$

$= 2(x^2 + 4)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

$x = \pm\sqrt{5}$

45. (a)



(b) $x = 4, 5, -5$

(c) $f(x) = x^3 - 4x^2 - 25x + 100$

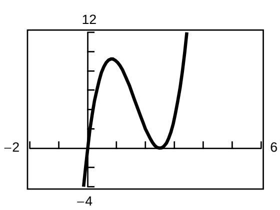
$= x^2(x - 4) - 25(x - 4)$

$= (x^2 - 25)(x - 4)$

$= (x - 5)(x + 5)(x - 4)$

$x = \pm 5, 4$

47. (a)



(b) x-intercepts: $(0, 0), \left(\frac{5}{2}, 0\right)$

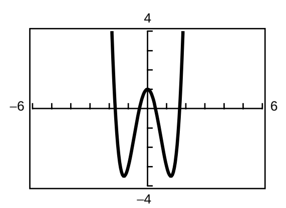
(c) $y = 4x^3 - 20x^2 + 25x$

$0 = 4x^3 - 20x^2 + 25x$

$0 = x(2x - 5)^2$

$x = 0$ or $x = \frac{5}{2}$

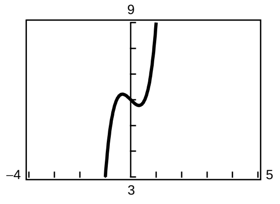
49.



Relative maximum: $(0, 1)$

Relative minimums: $(1.225, -3.5), (-1.225, -3.5)$

51.



Relative maximum: $(-0.324, 6.218)$

Relative minimum: $(0.324, 5.782)$

53. $f(x) = (x - 0)(x - 12)$

$f(x) = x^2 - 12x$

Note: $f(x) = a(x - 0)(x - 12) = ax(x - 12)$ has zeros 0 and 12 for all nonzero real numbers a.

55. $f(x) = (x - 2)(x - (-6))$

$= (x - 2)(x + 6)$

$= x^2 + 4x - 12$

Note: $f(x) = a(x - 2)(x + 6)$ has zeros 2 and -6 for all nonzero real numbers a.

57. $f(x) = (x - 0)[x - (-4)][x - (-3)]$

$= x(x + 4)(x + 3)$

$= x^3 + 7x^2 + 12x$

Note: $f(x) = ax(x + 4)(x + 3)$ has zeros $0, -4, -3$ for all nonzero real numbers a.

59. $f(x) = (x - 4)(x + 3)(x - 3)(x - 0)$

$= (x - 4)(x^2 - 9)x$

$= x^4 - 4x^3 - 9x^2 + 36x$

Note: $f(x) = a(x^4 - 4x^3 - 9x^2 + 36x)$ has these zeros for all nonzero real numbers a.

61. $f(x) = \left[x - \left(1 + \sqrt{3}\right)\right]\left[x - \left(1 - \sqrt{3}\right)\right]$

$= \left[(x - 1) - \sqrt{3}\right]\left[(x - 1) + \sqrt{3}\right]$

$= (x - 1)^2 - \left(\sqrt{3}\right)^2$

$= x^2 - 2x + 1 - 3$

$= x^2 - 2x - 2$

Note: $f(x) = a(x^2 - 2x - 2)$ has these zeros for all nonzero real numbers a.

63. $f(x) = (x - 2)\left[x - \left(4 + \sqrt{5}\right)\right]\left[x - \left(4 - \sqrt{5}\right)\right]$

$= (x - 2)\left[(x - 4) - \sqrt{5}\right]\left[(x - 4) + \sqrt{5}\right]$

$= (x - 2)[(x - 4)^2 - 5]$

$= x^3 - 10x^2 + 27x - 22$

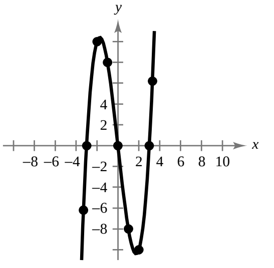
Note: $f(x) = a(x - 2)[(x - 4)^2 - 5]$ has these zeros for all nonzero real numbers a.

65. (a) The degree of f is odd and the leading coefficient is 1. The graph falls to the left and rises to the right.

(b) $f(x) = x^3 - 9x = x(x^2 - 9) = x(x - 3)(x + 3)$

zeros: $0, 3, -3$

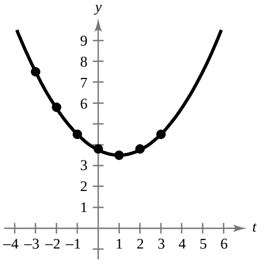
(c), (d)



67. (a) The degree of f is even and the leading coefficient is $\frac{1}{4}$. The graph rises to the left and to the right.

(b) $f(t) = \frac{1}{4}(t^2 - 2t + 15)$ has no real zeros.

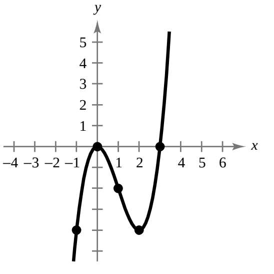
(c), (d)



69. (a) The degree of f is odd and the leading coefficient is 1. The graph falls to the left and rises to the right.

(b) $f(x) = x^3 - 3x^2 = x^2(x - 3)$; zeros: 0, 3

(c), (d)



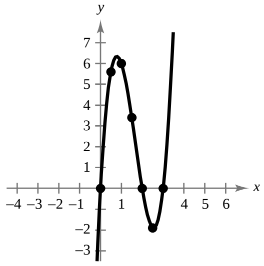
71. (a) The degree of f is odd and the leading coefficient is 3. The graph falls to the left and rises to the right.

(b) $f(x) = 3x^3 - 15x^2 + 18x = 3x(x^2 - 5x + 6)$

$= 3x(x - 2)(x - 3)$

zeros: 0, 2, 3

(c), (d)

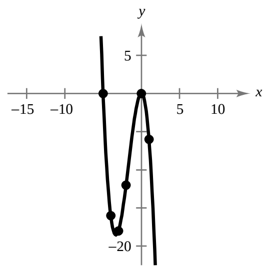


73. (a) The degree of f is odd and the leading coefficient is -1. The graph rises to the left and falls to the right.

(b) $f(x) = -x^3 - 5x^2 = x^2(-x - 5)$

zeros: $0, -5$

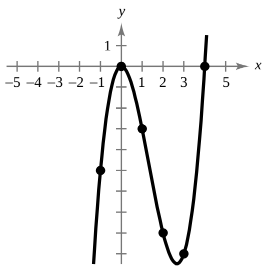
(c), (d)



75. (a) The degree of f is odd and the leading coefficient is 1. The graph falls to the left and rises to the right.

(b) $f(x) = x^2(x - 4)$; zeros: 0, 4

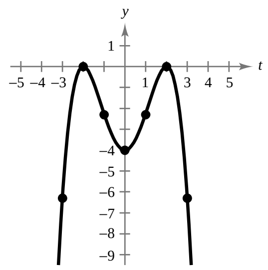
(c), (d)



77. (a) The degree of g is even (4) and the leading coefficient is $-\frac{1}{4}$. The graph falls to the left and to the right.

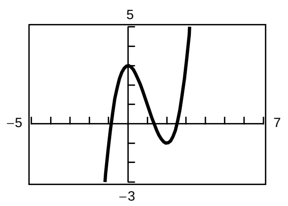
(b) $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)$; zeros: $2, -2$

(c)



79. $f(x) = x^3 - 3x^2 + 3$

(a)



The function has three zeros. They are in the intervals $(-1, 0)$, $(1, 2)$ and $(2, 3)$.

(b) $-0.879, 1.347, 2.532$

(c)

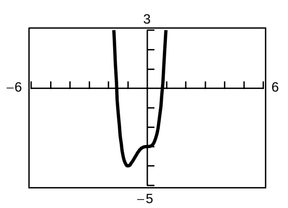
x	Y1
−0.9	−0.159
−0.89	−0.0813
−0.88	−0.0047
−0.87	−0.0708
−0.86	−0.14514
−0.85	−0.21838
−0.84	−0.2905

x	Y1
1.3	0.127
1.31	0.09979
1.32	0.07277
1.33	0.04594
1.34	0.0193
1.35	−0.0071
1.36	−0.0333

x	Y1
2.5	−0.125
2.51	−0.087
2.52	−0.0482
2.53	−0.0084
2.54	0.03226
2.55	0.07388
2.56	0.11642

81. $g(x) = 3x^4 + 4x^3 - 3$

(a)



The function has two zeros. They are in the intervals $(-2, -1)$ and $(0, 1)$.

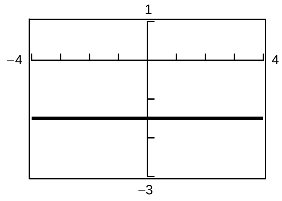
(b) $-1.585, 0.779$

(c)

x	Y1
−1.6	0.2768
−1.59	0.09515
−1.58	−0.0812
−1.57	−0.2524
−1.56	−0.4184
−1.55	−0.5795
−1.54	−0.7356

x	Y1
0.75	−0.3633
0.76	−0.2432
0.77	−0.1193
0.78	0.00866
0.79	0.14066
0.80	0.2768
0.81	0.41717

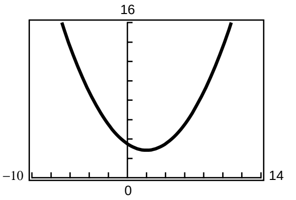
83. $f(x) = -\frac{3}{2}$



Horizontal line

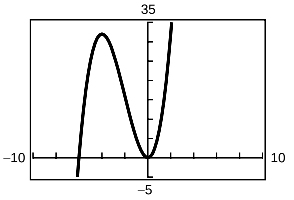
Xmin=-4
Xmax=4
Xscl=1
Ymin=-3
Ymax=1
Yscl=1

85. $f(t) = \frac{1}{6}(t^2 - 4t + 21)$



Xmin=-10
Xmax=14
Xscl=2
Ymin=0
Ymax=16
Yscl=2

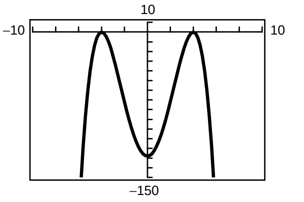
87.



No symmetry

Two x-intercepts

89.

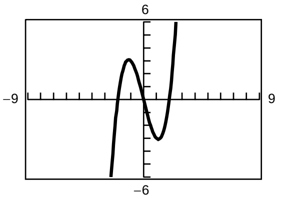


Symmetric about the y-axis

Two x-intercepts

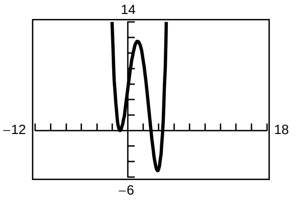
91. $f(x) = x^3 - 4x = x(x + 2)(x - 2)$

Symmetric to origin
Three x-intercepts



93. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

Three x-intercepts
No symmetry



95. (a) and (b)

Height, x	Length and Width	Volume, V
1	$36 - 2(1)$	$1[36 - 2(1)]^2 = 1156$
2	$36 - 2(2)$	$2[36 - 2(2)]^2 = 2048$
3	$36 - 2(3)$	$3[36 - 2(3)]^2 = 2700$
4	$36 - 2(4)$	$4[36 - 2(4)]^2 = 3136$
5	$36 - 2(5)$	$5[36 - 2(5)]^2 = 3380$
6	$36 - 2(6)$	$6[36 - 2(6)]^2 = 3456$
7	$36 - 2(7)$	$7[36 - 2(7)]^2 = 3388$

(c) Volume = length × width × height

Because the box is made from a square, length = width.

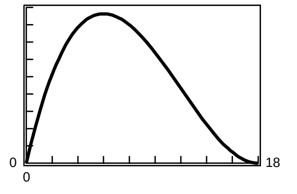
Thus:

$$\text{Volume} = (\text{length})^2 \times \text{height}$$
$$= (36 - 2x)^2 x$$

Domain: $0 < 36 - 2x < 36$

$$-36 < -2x < 0$$
$$18 > x > 0$$

(d)

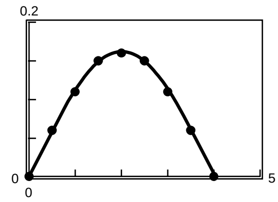


$x = 6$ when $V(x)$ is maximum.

97. The point of diminishing returns (where the graph changes from curving upward to curving downward) occurs when x = 200. The point is (200, 160) which corresponds to spending $2,000,000 on advertising to obtain a revenue of $160 million.

99. (a) $y = 0.003x^4 - 0.024x^3 + 0.020x^2 + 0.113x$

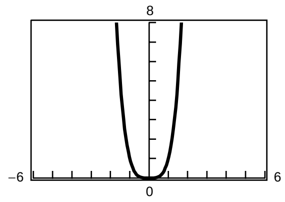
(b)



(c) The constant term should be zero. Yes, the model has zero as its constant term.

101. False. A fourth degree polynomial can have at most three turning points.

103. $f(x) = x^4$; $f(x)$ is even.



(a) $g(x) = f(x) + 2$

Vertical shift two units upward

$g(-x) = f(-x) + 2$

$= f(x) + 2$

$= g(x)$

Even

(b) $g(x) = f(x + 2)$

Horizontal shift two units to the left

Neither odd nor even

(c) $g(x) = f(-x) = (-x)^4 = x^4$

Reflection in the y-axis
The graph looks the same.
Even

(d) $g(x) = -f(x) = -x^4$

Reflection in the x-axis
Even

(e) $g(x) = f\left(\frac{1}{2}x\right) = \frac{1}{16}x^4$

Vertical shrink
Even

(f) $g(x) = \frac{1}{2}f(x) = \frac{1}{2}x^4$

Vertical shrink
Even

(g) $g(x) = f(x^{3/4}) = (x^{3/4}) = x^3$

Odd

(h) $g(x) = (f \circ f)(x) = f(f(x)) = f(x^4) = f(x^4)^4 = x^{16}$

Even

105. $(g - f)(3) = g(3) - f(3) = 8(3)^2 - [14(3) - 3]$

$= 72 - 39 = 33$

107. $\left(\frac{f}{g}\right)(-1.5) = \frac{f(-1.5)}{g(-1.5)} = \frac{-24}{18} = -\frac{4}{3}$

109. $(g \circ f)(0) = g(f(0)) = g(-3) = 8(-3)^2 = 72$