Section 2.4 Complex Numbers

- You should know how to work with complex numbers.
- Operations on complex numbers
  - (a) Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - (b) Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
  - (c) Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
  - (d) Division: $\dfrac{a + bi}{c + di} = \dfrac{a + bi}{c + di} \cdot \dfrac{c - di}{c - di} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}i$
- The complex conjugate of $a + bi$ is $a - bi$:
  $(a + bi)(a - bi) = a^2 + b^2$
- The additive inverse of $a + bi$ is $-a - bi$.
- The multiplicative inverse of $a + bi$ is
  $\dfrac{a - bi}{a^2 + b^2}.$
- $\sqrt{-a} = \sqrt{a}\,i$ for $a > 0$.

Solutions to Odd-Numbered Exercises

1. $a + bi = -9 + 4i$
$a = -9$
$b = 4$

3. $(a - 1) + (b + 3)i = 5 + 8i$
$a - 1 = 5 \implies a = 6$
$b + 3 = 8 \implies b = 5$

5. $4 + \sqrt{-25} = 4 + 5i$

7. $12 = 12 + 0i$

9. $-5i + i^2 = -5i - 1 = -1 - 5i$

11. $(\sqrt{-75})^2 = -75$

13. $\sqrt{-0.09} = \sqrt{0.09}\,i = 0.3i$

15. $(4 + i) + (7 - 2i) = 11 - i$

17. $(-1 + \sqrt{-8}) + (8 - \sqrt{-50}) = 7 + 2\sqrt{2}i - 5\sqrt{2}i = 7 - 3\sqrt{2}i$

19. $13i - (14 - 7i) = 13i - 14 + 7i = -14 + 20i$

21. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right) = -\frac{3}{2} - \frac{5}{2}i + \frac{5}{3} + \frac{11}{3}i$
$= -\frac{9}{6} - \frac{15}{6}i + \frac{10}{6} + \frac{22}{6}i$
$= \frac{1}{6} + \frac{7}{6}i$

23. $(1.6 + 3.2i) + (-5.8 + 4.3i) = -4.2 + 7.5i$

25. $\sqrt{-6} \cdot \sqrt{-2} = (\sqrt{6}i)(\sqrt{2}i) = \sqrt{12}i^2 = (2\sqrt{3})(-1) = -2\sqrt{3}$

27. $(\sqrt{-10})^2 = (\sqrt{10}i)^2 = 10i^2 = -10$

29. $(1 + i)(3 - 2i) = 3 - 2i + 3i - 2i^2$
$= 3 + i + 2$
$= 5 + i$

31. $6i(5 - 2i) = 30i - 12i^2 = 30i + 12 = 12 + 30i$

33. $(\sqrt{14} + \sqrt{10}\,i)(\sqrt{14} - \sqrt{10}\,i) = 14 - 10i^2 = 14 + 10 = 24$

35. $(4 + 5i)^2 = 16 + 40i + 25i^2 = 16 + 40i - 25$
$= -9 + 40i$

37. $\sqrt{-6}\sqrt{-6} = \sqrt{6}i\,\sqrt{6}i = 6i^2 = -6 \quad \left(\text{Note: } \sqrt{-6}\,\sqrt{-6} \neq \sqrt{(-6)(-6)}\right)$

39. $(4 + 3i)(4 - 3i) = 16 + 9 = 25$

41. $(-6 - \sqrt{5}i)(-6 + \sqrt{5}i) = 36 + 5 = 41$

43. $(22i)(-22i) = 484$

45. $(3 - \sqrt{-2})(3 + \sqrt{-2}) = (3 - \sqrt{2}i)(3 + \sqrt{2}i) = 9 + 2 = 11$

47. $\dfrac{6}{i} = \dfrac{6}{i} \cdot \dfrac{-i}{-i} = \dfrac{-6i}{-i^2} = \dfrac{-6i}{1} = -6i$

49. $\dfrac{4}{4 - 5i} = \dfrac{4}{4 - 5i} \cdot \dfrac{4 + 5i}{4 + 5i} = \dfrac{4(4 + 5i)}{16 + 25} = \dfrac{16 + 20i}{41} = \dfrac{16}{41} + \dfrac{20}{41}i$

51. $\dfrac{2 + i}{2 - i} = \dfrac{2 + i}{2 - i} \cdot \dfrac{2 + i}{2 + i} = \dfrac{4 + 4i + i^2}{4 + 1} = \dfrac{3 + 4i}{5} = \dfrac{3}{5} + \dfrac{4}{5}i$

53. $\dfrac{6 - 7i}{i} = \dfrac{6 - 7i}{i} \cdot \dfrac{-i}{-i} = \dfrac{-6i - 7}{1} = -7 - 6i$

55. $\dfrac{1}{(4 - 5i)^2} = \dfrac{1}{16 - 40i + 25i^2} = \dfrac{1}{-9 - 40i} \cdot \dfrac{-9 + 40i}{-9 + 40i}$
$= \dfrac{-9 + 40i}{81 + 1600} = \dfrac{-9 + 40i}{1681} = -\dfrac{9}{1681} + \dfrac{40}{1681}i$

57. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i} = \dfrac{2(1 - i) - 3(1 + i)}{(1 + i)(1 - i)}$
$= \dfrac{2 - 2i - 3 - 3i}{1 + 1}$
$= \dfrac{-1 - 5i}{2}$
$= -\dfrac{1}{2} - \dfrac{5}{2}i$

59. $-6i^3 + i^2 = -6i^2i + i^2$
$= -6(-1)i + (-1)$
$= 6i - 1$
$= -1 + 6i$

61. $-5i^5 = -5i^2i^2i$
$= -5(-1)(-1)i$
$= -5i$

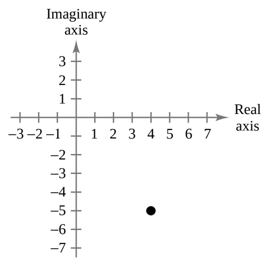
63. $(\sqrt{-75})^3 = (5\sqrt{3}i)^3 = 5^3(\sqrt{3})^3\, i^3$
$= 125(3\sqrt{3})(-i)$
$= -375\sqrt{3}i$

65. $\dfrac{1}{i^3} = \dfrac{1}{-i} = \dfrac{1}{-i} \cdot \dfrac{i}{i} = \dfrac{1}{-i^2} = \dfrac{i}{1} = i$

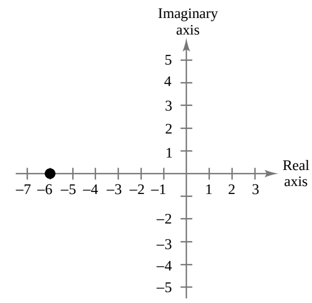
67. $4 + 3i$

69. $0 + 6i = 6i$

71. $4 - 5i$



73. -6



75. The complex number 0 is in the Mandelbrot Set since for $c = 0$, the corresponding Mandelbrot sequence is 0, 0, 0, 0, 0, 0, . . . which is bounded.

77. The complex number $\frac{1}{2}i$, is in the Mandelbrot Set since for $c = \frac{1}{2}i$, the corresponding Mandelbrot sequence is

$\frac{1}{2}i, -\frac{1}{4} + \frac{1}{2}i, -\frac{3}{16} + \frac{1}{4}i, -\frac{7}{256} + \frac{13}{32}i, \frac{-10{,}767}{65{,}536} + \frac{1957}{4096}i, -\frac{864{,}513{,}055}{4{,}294{,}967{,}296} + \frac{46{,}037{,}845}{134{,}217{,}728}i$

which is bounded. Or in decimal form

$0.5i, -0.25 + 0.5i, -0.1875 + 0.25i, -0.02734 + 0.40625i,$

$-0.164291 + 0.477783i, -0.201285 + 0.343009i.$

79. The complex number 1 is not in the Mandelbrot Set since for $c = 1$, the corresponding Mandelbrot sequence is 1, 2, 5, 26, 677, 458,330 which is unbounded.

81. $(2)^3 = 8$

$$\begin{aligned}
\left(-1 + \sqrt{3}i\right)^3 &= (-1)^3 + 3(-1)^2\left(\sqrt{3}i\right) + 3(-1)\left(\sqrt{3}i\right)^2 + \left(\sqrt{3}i\right)^3 \\
&= -1 + 3\sqrt{3}i - 9i^2 + 3\sqrt{3}i^3 \\
&= -1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i \\
&= 8
\end{aligned}$$

$$\begin{aligned}
\left(-1 - \sqrt{3}i\right)^3 &= (-1)^3 + 3(-1)^2\left(-\sqrt{3}i\right) + 3(-1)\left(-\sqrt{3}i\right)^2\left(\sqrt{3}i\right)^3 \\
&= -1 - 3\sqrt{3}i - 9i^2 - 3\sqrt{3}i^3 \\
&= -1 - 3\sqrt{3}i + 9 + 3\sqrt{3}i \\
&= 8
\end{aligned}$$

The three numbers are cube roots of 8.

83. (a) $z_1 = 5 + 2i$

$z_2 = 3 - 4i$

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{5 + 2i} + \frac{1}{3 - 4i} \\
&= \frac{(3 - 4i) + (5 + 2i)}{(5 + 2i)(3 - 4i)} \\
&= \frac{8 - 2i}{23 - 14i} \\
z &= \frac{23 - 14i}{8 - 2i}\left(\frac{8 + 2i}{8 + 2i}\right) \\
&= \frac{212 - 66i}{68} \approx 3.118 - 0.971i
\end{aligned}$$

(b) $z_1 = 16i + 9$

$z_2 = 20 - 10i$

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{9 + 16i} + \frac{1}{20 - 10i} \\
&= \frac{(20 - 10i) + (9 + 16i)}{(9 + 16i)(20 - 10i)} \\
&= \frac{29 + 6i}{340 + 230i} \\
z &= \frac{340 + 230i}{29 + 6i}\left(\frac{29 - 6i}{29 - 6i}\right) \\
&= \frac{11240 + 4630i}{877} \approx 12.816 + 5.279i
\end{aligned}$$