Section 2.6 Rational Functions and Asymptotes

- You should know the following basic facts about rational functions.
  - (a) A function of the form $f(x) = P(x)/Q(x)$, $Q(x) \neq 0$, where $P(x)$ and $Q(x)$ are polynomials, is called a rational function.
  - (b) The domain of a rational function is the set of all real numbers except those which make the denominator zero.
  - (c) If $f(x) = P(x)/Q(x)$ is in reduced form, and a is a value such that $Q(a) = 0$, then the line $x = a$ is a vertical asymptote of the graph of f. $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$.
  - (d) The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \longrightarrow b$ as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.
  - (e) Let $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}$ where $P(x)$ and $Q(x)$ have no common factors.
    1. If $n < m$, then the x-axis ($y = 0$) is a horizontal asymptote.
    2. If $n = m$, then $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote.
    3. If $n > m$, then there are no horizontal asymptotes.

Solutions to Odd-Numbered Exercises

1. $f(x) = \dfrac{1}{x - 1}$

(a)

x	$f(x)$
0.5	-2
0.9	-10
0.99	-100
0.999	-1000

x	$f(x)$
1.5	2
1.1	10
1.01	100
1.001	1000

x	$f(x)$
5	0.25
10	$0.\overline{1}$
100	$0.\overline{01}$
1000	$0.\overline{001}$

x	$f(x)$
-5	-0.167
-10	-0.0909
-100	-0.0099
-1000	-0.001

(b) The zero of the denominator is $x = 1$, so $x = 1$ is a vertical asymptote. The degree of the numerator is less than the degree of the denominator so the x-axis, or $y = 0$ is a horizontal asymptote.

(c) The domain is all real numbers except $x = 1$.

3. $f(x) = \dfrac{3x}{|x - 1|}$

(a)

x	$f(x)$
0.5	3
0.9	27
0.99	297
0.999	2997

x	$f(x)$
1.5	9
1.1	33
1.01	303
1.001	3003

x	$f(x)$
5	3.75
10	$3.\overline{33}$
100	$3.\overline{03}$
1000	$3.\overline{003}$

x	$f(x)$
-5	-2.5
-10	-2.727
-100	-2.970
-1000	-2.997

(b) The zero of the denominator is $x = 1$, so $x = 1$ is a vertical asymptote. Since $f(x) \longrightarrow 3$ as $x \longrightarrow \infty$ and $f(x) \longrightarrow -3$ as $x \longrightarrow -\infty$, both $y = 3$ and $y = -3$ are horizontal asymptotes.

(c) The domain is all real numbers except $x = 1$.

5. $f(x) = \dfrac{3x^2}{x^2 - 1}$

(a)

x	$f(x)$
0.5	-1
0.9	-12.79
0.99	-148.79
0.999	-1498

x	$f(x)$
1.5	5.4
1.1	17.29
1.01	152.3
1.001	1502.3

x	$f(x)$
5	3.125
10	$3.\overline{03}$
100	$3.\overline{0003}$
1000	3

x	$f(x)$
-5	3.125
-10	$3.\overline{03}$
-100	$3.\overline{0003}$
-1000	3

(b) The zeros of the denominator are $x = \pm 1$ so both $x = 1$ and $x = -1$ are vertical asymptotes.

Since the degree of the numerator equals the degree of the denominator, $y = \frac{3}{1} = 3$ is a horizontal asymptote.

(c) The domain is all real numbers except $x = \pm 1$.

7. $f(x) = \dfrac{2}{x + 2}$

Vertical asymptote: $x = -2$

Horizontal asymptote: $y = 0$

Matches graph (a)

9. $f(x) = \dfrac{4x + 1}{x}$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 4$

Matches graph (c)

11. $f(x) = \dfrac{x - 2}{x - 4}$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 1$

Matches graph (b)

13. $f(x) = \dfrac{1}{x^2}$

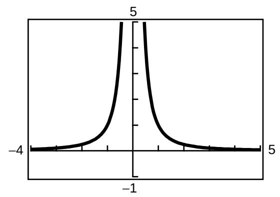
(a) Domain: all real numbers except $x = 0$

(b) Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

[Degree of $p(x)$ < degree of $q(x)$]

(c)



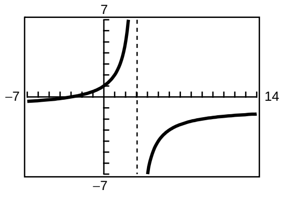
15. $f(x) = \dfrac{3 + x}{3 - x}$

(a) Domain: all real numbers except $x = 3$.

(b) Vertical asymptote: $x = 3$

Horizontal asymptote: $y = -1$

(c)



17. $f(x) = \dfrac{2x^3}{x^2 - 1}$

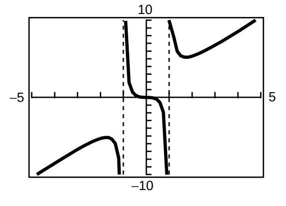
(a) Domain: all real numbers except $x = \pm 1$

(b) Vertical asymptotes: $x = \pm 1$

Horizontal asymptotes: None

[Degree of $p(x)$ > degree of $q(x)$]

(c)



19. $f(x) = \dfrac{x^2 - 4}{x + 2}$, $g(x) = x - 2$

(a) Domain of *f*: all real numbers except -2
Domain of *g*: all real numbers

(b) Since $x + 2$ is a common factor of both the numerator and the denominator of $f(x)$, $x = -2$ is not a vertical asymptote of *f*. *f* has no vertical asymptotes.

(c)

x	-4	-3	-2.5	-2	-1.5	-1	0
$f(x)$	-6	-5	-4.5	undef.	-3.5	-3	-2
$g(x)$	-6	-5	-4.5	-4	-3.5	-3	-2

(d) *f* and *g* differ only where *f* is undefined.

21. $f(x) = \dfrac{x - 3}{x^2 - 3x}$, $g(x) = \dfrac{1}{x}$

(a) Domain of *f*: all real number except 0 and 3
Domain of *g*: all real numbers except 0

(b) Since $x - 3$ is a common factor of both the numerator and the denominator of *f*, $x = 3$ is not a vertical asymptote of *f*. The only vertical asymptote is $x = 0$.

(c)

x	-1	-0.5	0	0.5	2	3	4
$f(x)$	-1	-2	undef.	2	$\frac{1}{2}$	undef.	$\frac{1}{4}$
$g(x)$	-1	-2	undef	2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

(d) They differ only at $x = 3$, where *f* is undefined and *g* is defined.

23. $f(x) = 4 - \dfrac{1}{x}$

(a) As $x \longrightarrow \pm\infty$, $f(x) \longrightarrow 4$

(b) As $x \longrightarrow \infty$, $f(x) \longrightarrow 4$ but is less than 4

(c) As $x \longrightarrow -\infty$, $f(x) \longrightarrow 4$ but is greater than 4

25. $f(x) = \dfrac{2x - 1}{x - 3}$

(a) As $x \longrightarrow \pm\infty$, $f(x) \longrightarrow 2$

(b) As $x \longrightarrow \infty$, $f(x) \longrightarrow 2$ but is greater than 2

(c) As $x \longrightarrow -\infty$, $f(x) \longrightarrow 2$ but is less than 2

27. $f(x) = \dfrac{x^2 - 9}{x + 1} = \dfrac{(x + 3)(x - 3)}{x + 1}$

The zeros of *f* correspond to the zeros of the numerator and are $x = \pm 3$.

29. $f(x) = 1 - \dfrac{2}{x - 5} = \dfrac{x - 7}{x - 5}$

The zero of *f* corresponds to the zero of the numerator and is $x = 7$.

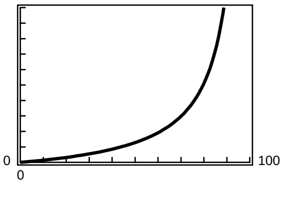
31. $C = \dfrac{255p}{100 - p}$, $0 \le p < 100$

(a) $C(10) = \dfrac{255(10)}{100 - 10} \approx 28.33$ million dollars

(b) $C(40) = \dfrac{255(40)}{100 - 40} = 170$ million dollars

(c) $C(75) = \dfrac{255(75)}{100 - 75} = 765$ million dollars

(d)



(e) $C \longrightarrow \infty$ as $x \longrightarrow 100$. No, it would not be possible to remove 100% of the pollutants.

33. (a)

M	200	400	600	800	1000	1200	1400	1600	1800	2000
t	0.472	0.596	0.710	0.817	0.916	1.009	1.096	1.178	1.255	1.328

The greater the mass, the more time required per oscillation. The model is a good fit to the actual data.

(b) You can find M corresponding to $t = 1.056$ by finding the point of intersection of

$$t = \frac{38M + 16{,}965}{10(M + 500)} \quad \text{and} \quad t = 1.056.$$

If you do this, you obtain $M \approx 1306$ grams.

35. $N = \dfrac{20(5 + 3t)}{1 + 0.04t},\ 0 \le t$

(a) $N(5) \approx 333$ deer

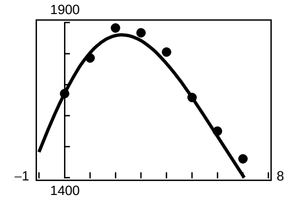
$N(10) = 500$ deer

$N(25) = 800$ deer

(b) The herd is limited by the horizontal asymptote:

$$N = \frac{60}{0.04} = 1500 \text{ deer}$$

37. (a)



(b) For 2002, $t = 12$ and $M \approx 889$ thousand

(c) No, this model predicts that $M \to 0$ as t increases.

39. True, $f(x) = x^3 - 2x^2 - 5x + 6 = \dfrac{x^3 - 2x^2 - 5x + 6}{1}$ is a rational function.

41. $f(x) = \dfrac{1}{x^2 + 1}$ is one possible answer.

43. $f(x) = \dfrac{-3x^2}{x(2x - 5)} = \dfrac{-3x^2}{2x^2 - 5x}$ is one possible answer.

45.
$$\begin{aligned} x(10 - x) &= 25 \\ 0 &= x^2 - 10x + 25 \\ 0 &= (x - 5)^2 \\ x &= 5 \end{aligned}$$

47.
$$\begin{aligned} t^3 - 50t &= 0 \\ t(t^2 - 50) &= 0 \\ t &= 0,\ \pm 5\sqrt{2} \end{aligned}$$

49.
$$\begin{aligned} x^4 - 225 &= 0 \\ (x^2 - 15)(x^2 + 15) &= 0 \\ x &= \pm\sqrt{15},\ \pm\sqrt{15}\, i \end{aligned}$$

51.
$$\begin{array}{r|rrr} 3 & 1 & -10 & 15 \\ & & 3 & -21 \\ \hline & 1 & -7 & -6 \end{array}$$

$$\frac{x^2 - 10x + 15}{x - 3} = x - 7 + \frac{-6}{x - 3}$$

53.
$$\begin{array}{r|rrr} -6 & 4 & 3 & -10 \\ & & -24 & 126 \\ \hline & 4 & -21 & 116 \end{array}$$

$$\frac{4x^2 + 3x - 10}{x + 6} = 4x - 21 + \frac{116}{x + 6}$$

55. $(x + 2)(x - 6i)(x + 6i) = (x + 2)(x^2 + 36) = x^3 + 2x^2 + 36x + 72$

57.
$$\begin{aligned} (x - 1)(x - (-3 + 2i))(x - (-3 - 2i)) &= (x - 1)((x + 3)^2 + 4) \\ &= (x - 1)(x^2 + 6x + 13) \\ &= x^3 + 5x^2 + 7x - 13 \end{aligned}$$