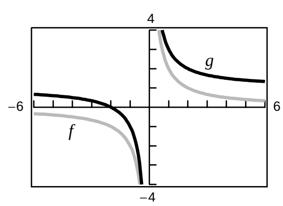
Section 2.7 Graphs of Rational Functions

- You should be able to graph $f(x) = \dfrac{p(x)}{q(x)}$.
  - (a) Find the x- and y-intercepts.
  - (b) Find any vertical or horizontal asymptotes.
  - (c) Plot additional points.
  - (d) If the degree of the numerator is one more than the degree of the denominator, use long division to find the slant asymptote.

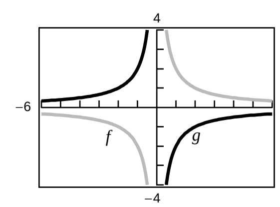
Solutions to Odd-Numbered Exercises

1. $g(x) = \dfrac{2}{x} + 1$



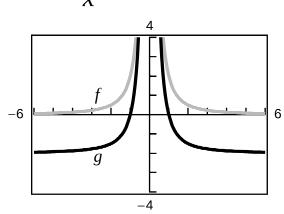
Vertical shift one unit upward

3. $g(x) = -\dfrac{2}{x}$



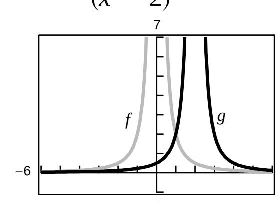
Reflection in the x-axis

5. $g(x) = \dfrac{2}{x^2} - 2$



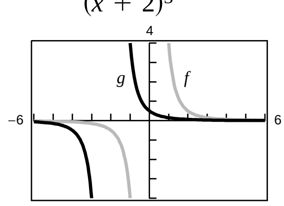
Vertical shift two units downward

7. $g(x) = \dfrac{2}{(x - 2)^2}$



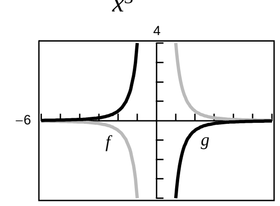
Horizontal shift two units to the right

9. $g(x) = \dfrac{4}{(x + 2)^3}$



Horizontal shift two units to the left

11. $g(x) = -\dfrac{4}{x^3}$



Reflection in the x-axis

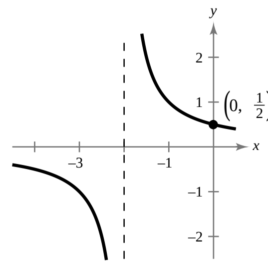
13. $f(x) = \dfrac{1}{x+2}$

y-intercept: $\left(0, \dfrac{1}{2}\right)$

Vertical asymptote: $x = -2$

Horizontal asymptote: $y = 0$

x	-4	-3	-1	0	1
y	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$	$\frac{1}{3}$



15. $C(x) = \dfrac{5+2x}{1+x} = \dfrac{2x+5}{x+1}$

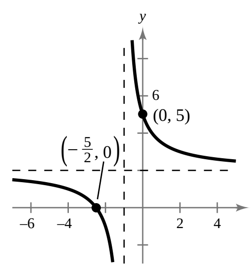
x-intercept: $\left(-\dfrac{5}{2}, 0\right)$

y-intercept: $(0, 5)$

Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 2$

x	-4	-3	-2	0	1	2
$C(x)$	1	$\frac{1}{2}$	-1	5	$\frac{7}{2}$	3



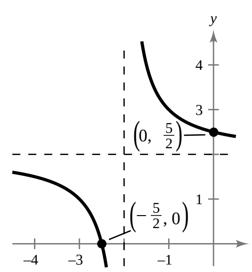
17. $g(x) = \dfrac{1}{x+2} + 2 = \dfrac{2x+5}{x+2}$

Intercepts: $\left(-\dfrac{5}{2}, 0\right), \left(0, \dfrac{5}{2}\right)$

Vertical asymptote: $x = -2$

Horizontal asymptote: $y = 2$

x	-4	-3	-1	0	1
y	$\frac{3}{2}$	1	3	$\frac{5}{2}$	$\frac{7}{3}$



Note: This is the graph of $f(x) = \dfrac{1}{x+2}$ (Exercise 13) shifted upward two units.

19. $f(x) = 2 - \dfrac{3}{x^2} = \dfrac{2x^2 - 3}{x^2}$

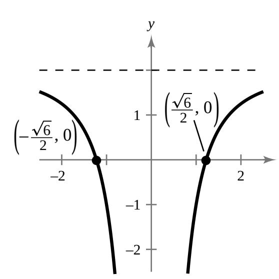
x-intercepts: $\left(-\dfrac{\sqrt{6}}{2}, 0\right), \left(\dfrac{\sqrt{6}}{2}, 0\right)$

Vertical asymptote: $x = 0$

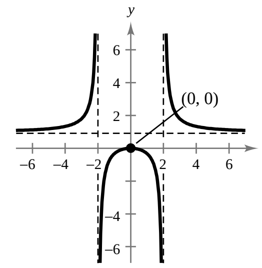
Horizontal asymptote: $y = 2$

y-axis symmetry

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{5}{4}$	-1	-10	-10	-1	$\frac{5}{4}$



21. $f(x) = \dfrac{x^2}{x^2 - 4}$



Intercept: $(0, 0)$

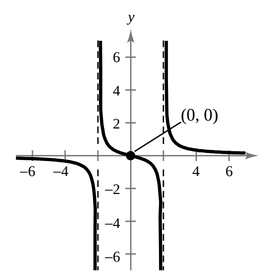
Vertical asymptotes: $x = 2,\ x = -2$

Horizontal asymptote: $y = 1$

y-axis symmetry

x	± 4	± 3	± 1	0
y	$\pm\frac{4}{3}$	$\frac{9}{5}$	$-\frac{1}{3}$	0

23. $f(x) = \dfrac{x}{x^2 - 4}$



Intercepts: $(0, 0)$

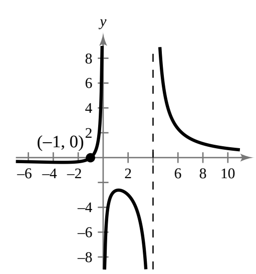
Vertical asymptotes: $x = 2,\ x = -2$

Horizontal asymptote: $y = 0$

Origin symmetry

x	-4	-3	-1	0	1	3	4
y	$-\frac{1}{3}$	$-\frac{3}{5}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{3}{5}$	$\frac{1}{3}$

25. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$



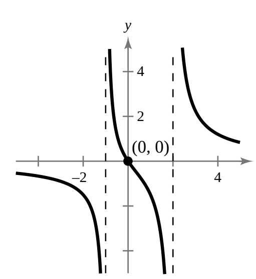
Intercept: $(-1, 0)$

Vertical asymptotes: $x = 0$ and $x = 4$

Horizontal asymptote: $y = 0$

x	-2	-1	1	2	3	5	6
y	$-\frac{1}{3}$	0	$-\frac{8}{3}$	-3	$-\frac{16}{3}$	$\frac{24}{5}$	$\frac{7}{3}$

27. $f(x) = \dfrac{3x}{x^2 - x - 2} = \dfrac{3x}{(x + 1)(x - 2)}$



Intercept: $(0, 0)$

Vertical asymptotes: $x = -1, 2$

Horizontal asymptote: $y = 0$

x	-3	0	1	3	4
y	$-\frac{9}{10}$	0	$-\frac{3}{2}$	$\frac{9}{4}$	$\frac{6}{5}$

29. $f(x) = \dfrac{-4}{\dfrac{5}{x} - 5} = \dfrac{-4}{\dfrac{5 - 5x}{x}} = \dfrac{-4x}{5 - 5x} = \dfrac{4}{5}\,\dfrac{x}{x - 1},\ x \neq 0$

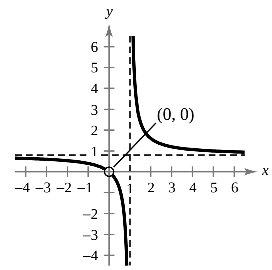
No intercepts [Note: $f(0)$ is not defined]

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = \dfrac{4}{5}$

No symmetry

x	-2	-1	0	1	2	3
y	$\frac{8}{15}$	$\frac{2}{5}$	undef.	undef.	$\frac{8}{5}$	$\frac{6}{5}$



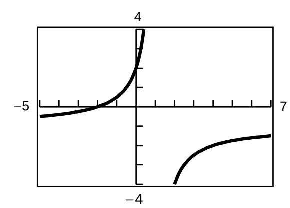
31. $f(x) = \dfrac{2 + x}{1 - x} = -\dfrac{x + 2}{x - 1}$

x-intercept: $(-2, 0)$

y-intercept: $(0, 2)$

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = -1$



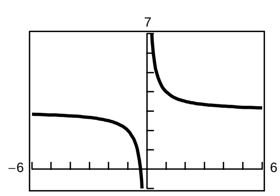
Domain: $x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$

33. $f(t) = \dfrac{3t + 1}{t}$

t-intercept: $\left(-\dfrac{1}{3}, 0\right)$

Vertical asymptote: $t = 0$

Horizontal asymptote: $y = 3$

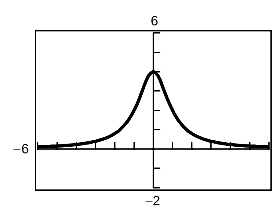


Domain: $t \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

35. $h(t) = \dfrac{4}{t^2 + 1}$

Domain: all real numbers OR $(-\infty, \infty)$

Horizontal asymptote: $y = 0$

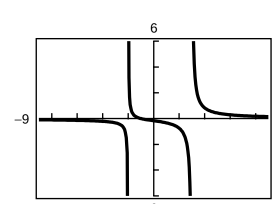


37. $f(x) = \dfrac{x + 1}{x^2 - x - 6} = \dfrac{x + 1}{(x - 3)(x + 2)}$

Domain: all real numbers except $x = 3, -2$

Vertical asymptotes: $x = 3,\ x = -2$

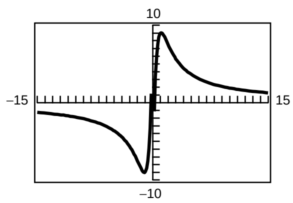
Horizontal asymptote: $y = 0$



39. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$

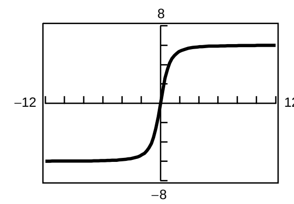
Domain: all real numbers except 0,
OR $(-\infty, 0) \cup (0, \infty)$

Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$
Origin Symmetry

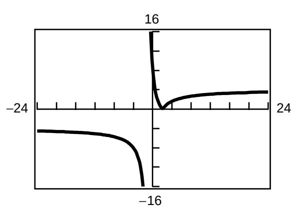


41. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$

There are two horizontal asymptotes: $y = \pm 6$



43. $g(x) = \dfrac{4|x - 2|}{x + 1}$

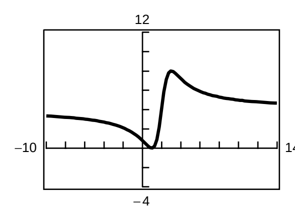


There are two horizontal asymptotes: $y = \pm 4$.

One vertical asymptote: $x = -1$.

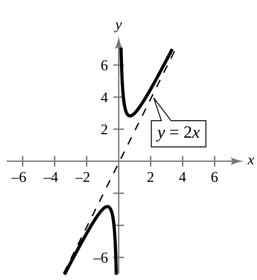
45. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$

The graph crosses its horizontal asymptote: $y = 4$



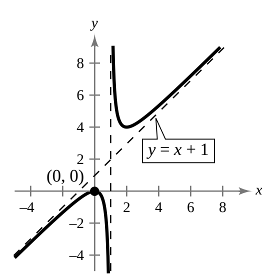
47. $f(x) = \dfrac{2x^2 + 1}{x} = 2x + \dfrac{1}{x}$

Vertical asymptote: $x = 0$
Slant asymptote: $y = 2x$
Origin symmetry



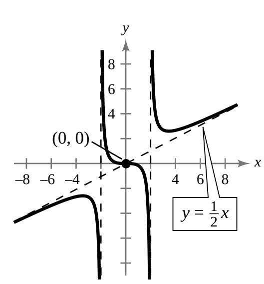
49. $h(x) = \dfrac{x^2}{x - 1} = x + 1 + \dfrac{1}{x - 1}$

Intercept: $(0, 0)$
Vertical asymptote: $x = 1$
Slant asymptote: $y = x + 1$



51. $g(x) = \dfrac{x^3}{2x^2 - 8} = \dfrac{1}{2}x + \dfrac{4x}{2x^2 - 8}$

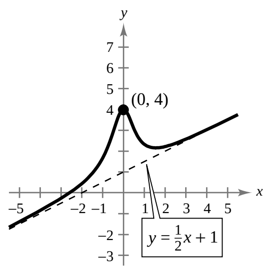
Intercept: $(0, 0)$
Vertical asymptotes: $x = \pm 2$
Slant asymptote: $y = \dfrac{1}{2}x$
Origin symmetry



53. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1} = \dfrac{x}{2} + 1 + \dfrac{3 - \dfrac{x}{2}}{2x^2 + 1}$

Intercepts: $(-2.594, 0), (0, 4)$

Slant asymptote: $y = \dfrac{x}{2} + 1$



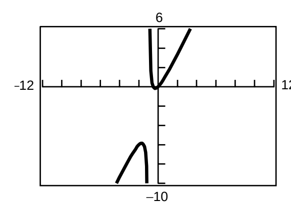
55. (a) x-intercept: $(-1, 0)$

(b) $$0 = \frac{x+1}{x-3}$$
$$0 = x + 1$$
$$-1 = x$$

57. (a) x-intercepts: $(\pm 1, 0)$

(b) $$0 = \frac{1}{x} - x$$
$$x = \frac{1}{x}$$
$$x^2 = 1$$
$$x = \pm 1$$

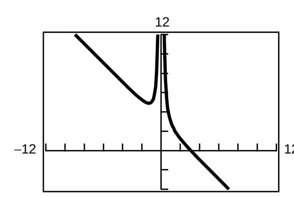
59. $y = \dfrac{2x^2 + x}{x + 1} = 2x - 1 + \dfrac{1}{x + 1}$



Domain: all real numbers except $x = -1$
Vertical asymptote: $x = -1$
Slant asymptote: $y = 2x - 1$

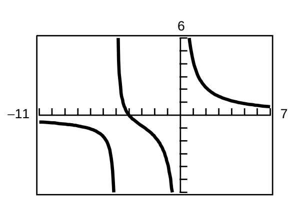
61. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2} = \dfrac{1}{x^2} + 3 - x = -x + 3 + \dfrac{1}{x^2}$

Domain:all real numbers except 0
OR $(-\infty, 0) \cup (0, \infty)$
Vertical asymptote: $x = 0$
Slant asymptote: $y = -x + 3$



63. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$

(a)

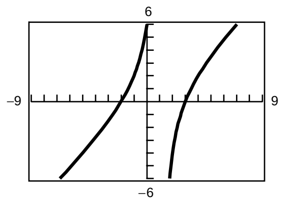


x-intercept: $(-4, 0)$

(b) $$0 = \frac{1}{x+5} + \frac{4}{x}$$
$$-\frac{4}{x} = \frac{1}{x+5}$$
$$-4(x + 5) = x$$
$$-4x - 20 = x$$
$$-5x = 20$$
$$x = -4$$

65. $y = x - \dfrac{6}{x-1}$

(a)



x-intercept: $(-2, 0), (3, 0)$

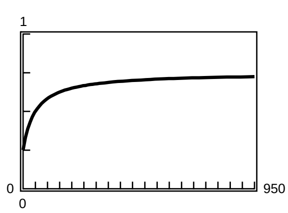
(b)
$$0 = x - \frac{6}{x-1}$$
$$\frac{6}{x-1} = x$$
$$6 = x(x-1)$$
$$0 = x^2 - x - 6$$
$$0 = (x+2)(x-3)$$
$$x = -2, \quad x = 3$$

67. (a) $0.25(10) + 0.75(x) = C(10 + x)$

$$C = \frac{2.5 + 0.75x}{10 + x} \cdot \frac{4}{4}$$
$$C = \frac{10 + 3x}{4(10 + x)}$$
$$= \frac{3x + 10}{4(x + 10)}$$

(b) Domain: $x > 0$ and $x \le 1000 - 10$
Thus, $0 \le x \le 990$ OR $[0, 990]$.

(c)



As the tank is filled, the rate that the concentration is increasing slows down. It approaches the horizontal asymptote of $C = \frac{3}{4} = 0.75$. The concentration reaches 74.5% when the tank is full ($x = 990$).

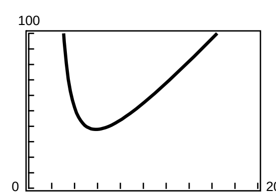
69. (a) $A = xy$ and

$$(x-2)(y-4) = 30$$
$$y - 4 = \frac{30}{x-2}$$
$$y = 4 + \frac{30}{x-2} = \frac{4x+22}{x-2}$$

Thus, $A = xy = x\left(\dfrac{4x+22}{x-2}\right) = \dfrac{2x(2x+11)}{x-2}$.

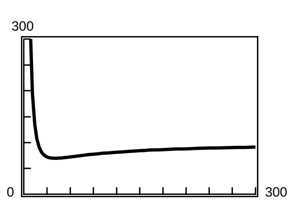
(b) Domain: Since the margins on the left and right are each 1 inch, $x > 2$, OR $(2, \infty)$.

(c)



The area is minimum when $x \approx 5.87$ in. and $y \approx 11.75$ in.

71. $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x+30}\right),\ 1 \le x$

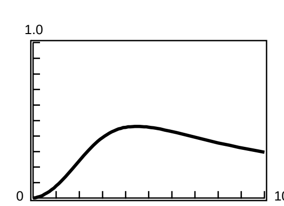


The minimum occurs when $x \approx 40.4 \approx 40$.

73. $C = \dfrac{3t^2 + t}{t^3 + 50},\ 0 \le t$

(a) The horizontal asymptote is the t-axis, or $C = 0$. This indicates that the chemical eventually dissipates.

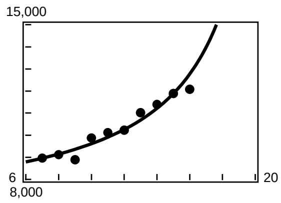
(b)



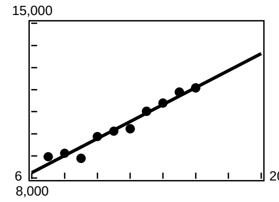
The maximum occurs when $t \approx 4.5$.

(c) $C < 0.345$ when $0 \le t < 2.65$ hours and when $t > 8.32$ hours

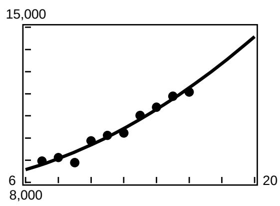
75. (a)



(b) $K = 384.49t + 5937.65$



(c) $K = 14.87t^2 + 42.54t + 7781.22$



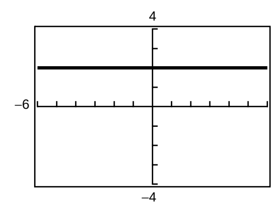
(d) Quadratic model is best because it fits the data well and is easy to use. Answer will vary.

77. False, you will have to lift your pencil to cross the vertical asymptote.

79. $h(x) = \dfrac{6 - 2x}{3 - x} = \dfrac{2(3 - x)}{3 - x}$

Since $h(x)$ is not reduced and $(3 - x)$ is a factor of both the numerator and the denominator, $x = 3$ is not a horizontal asymptote.

There is a hole in the graph at $x = 3$.



81. No, there are rational functions without vertical asymptotes. Two examples are

$f(x) = \dfrac{6 - 2x}{3 - x}$ (see Exercise 79)

$g(x) = \dfrac{1}{x^2 + 1}$

83. $y = x - 2 + \dfrac{a}{x + 4}$ has slant asymptote $y = x - 2$ and vertical asymptote at $x = -4$.

We determine a so that y has a zero at $x = 3$:

$$0 = 3 - 2 + \frac{a}{3 + 4} = 1 + \frac{a}{7} \Longrightarrow a = -7$$

Hence, $y = x - 2 + \dfrac{-7}{x + 4} = \dfrac{x^2 + 2x - 15}{x + 4}$

85. $y = \dfrac{-2(x + 6)}{(x - 3)}$ has vertical asymptote $x = 3$,

horizontal asymptote $y = -2$ and zero at $x = -6$

87. $4x + 5y - 2 = 0 \Longrightarrow y = \dfrac{1}{5}(-4x + 2) = -\dfrac{4}{5}x + \dfrac{2}{5}$ line