CHAPTER 3
Exponential and Logarithmic Functions

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Section 3.1 Exponential Functions and Their Graphs

- You should know that a function of the form $y = a^x$, where $a > 0$, $a \neq 1$, is called an exponential function with base a.
- You should be able to graph exponential functions.
- You should be familiar with the number e and the natural exponential function $f(x) = e^x$.
- You should know formulas for compound interest.
  (a) For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$.
  (b) For continuous compoundings: $A = Pe^{rt}$.

Solutions to Odd-Numbered Exercises

1. $(3.4)^{6.8} \approx 4112.033$

3. $6^{2\pi} \approx 77{,}494.076$

5. $\sqrt[3]{7493} \approx 19.568$

7. $e^{1/2} \approx 1.649$

9. $e^{9.2} \approx 9897.129$

11. $$\begin{aligned} f(x) &= 3^{x-2} \\ &= 3^x 3^{-2} \\ &= 3^x\left(\frac{1}{3^2}\right) \\ &= \frac{1}{9}(3^x) \\ &= h(x) \end{aligned}$$

Thus, $f(x) \neq g(x)$, but $f(x) = h(x)$. You can confirm your answer graphically by graphing f, g, and h in the same viewing rectangle.

13. $$\begin{aligned} f(x) &= 16(4^{-x}) \\ &= 4^2(4^{-x}) \\ &= 4^{2-x} \\ &= \left(\frac{1}{4}\right)^{-(2-x)} \\ &= \left(\frac{1}{4}\right)^{x-2} \\ &= g(x) \end{aligned}$$ and $$\begin{aligned} f(x) &= 16(4^{-x}) \\ &= 16(2^2)^{-x} \\ &= 16(2^{-2x}) \\ &= h(x) \end{aligned}$$

Thus, $f(x) = g(x) = h(x)$. You can confirm your answer graphically by graphing f, g, and h in the same viewing rectangle.

15. $f(x) = 2^x$ rises to the right.
Asymptote: $y = 0$
Intercept: $(0, 1)$
Matches graph (c).

17. $f(x) = 2^{-x}$ falls to the right.
Asymptote: $y = 0$
Intercept: $(0, 1)$
Matches graph (e).

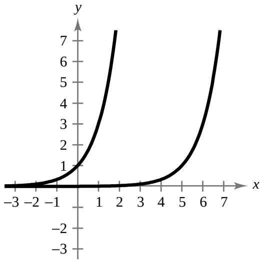
19. $f(x) = 2^x - 4$ rises to the right.
Asymptote: $y = -4$
Intercept: $(0, -3)$
Matches graph (g).

21. $f(x) = -2^{x-2} = -(2^{x-2})$ falls to the right.
Asymptote: $y = 0$
Intercept: $(0, -2^{-2}) = \left(0, -\frac{1}{4}\right)$
Matches graph (a).

23. $f(x) = 3^x$

$g(x) = 3^{x-5} = f(x - 5)$

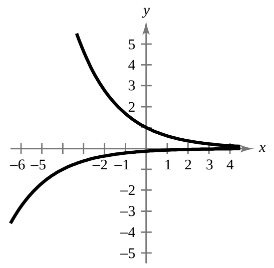
Horizontal shift five units to the right



25. $f(x) = \left(\frac{3}{5}\right)^x$

$g(x) = -\left(\frac{3}{5}\right)^{x+4} = -f(x + 4)$

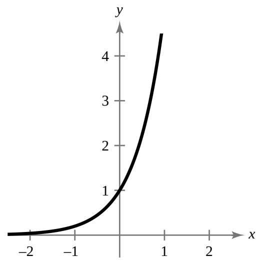
Horizontal shift 4 units to the left, followed by reflection in x-axis.



27. $g(x) = 5^x$

x	-2	-1	0	1	2
$g(x)$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25

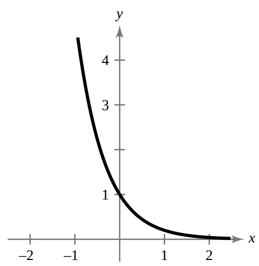
(a) Asymptote: $y = 0$
(b) Intercept: $(0, 1)$
(c) Increasing



29. $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$

x	-2	-1	0	1	2
y	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$

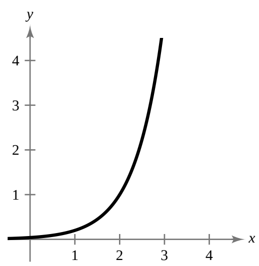
(a) Asymptote: $y = 0$
(b) Intercepts: $(0, 1)$
(c) Decreasing



31. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5

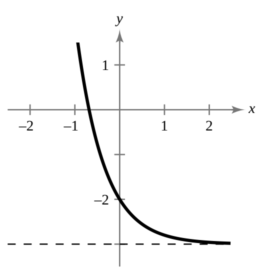
(a) Asymptote: $y = 0$
(b) Intercepts: $\left(0, \frac{1}{25}\right)$
(c) Increasing



33. $g(x) = 5^{-x} - 3$

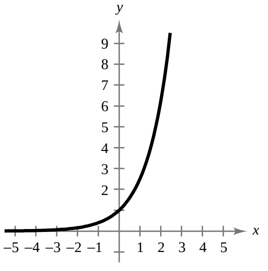
x	-1	0	1	2
y	2	-2	$-2\frac{4}{5}$	$-2\frac{24}{25}$

(a) Asymptote: $y = -3$
(b) Intercepts: $(0, -2)$, $(-0.683, 0)$
(c) Decreasing



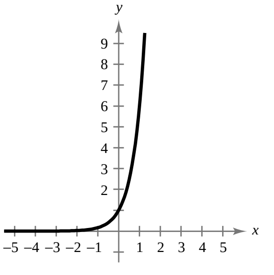
35. $f(x) = \left(\frac{5}{2}\right)^x$

x	-1	0	1	2	3
$f(x)$	0.4	1	2.5	6.25	15.625



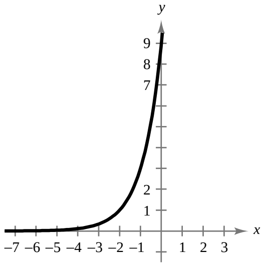
37. $f(x) = 6^x$

x	-1	0	1	2
$f(x)$	0.167	1	6	36



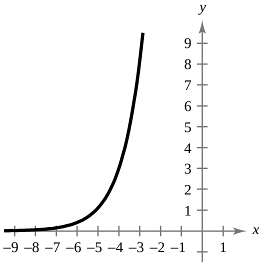
39. $f(x) = 3^{x+2} = 9 \cdot 3^x$

x	-3	-2	0	1
$f(x)$	$\frac{1}{3}$	1	9	27



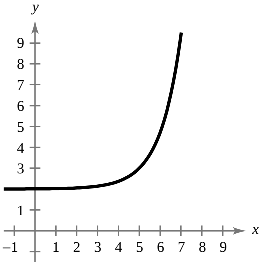
41. $f(x) = 3e^{x+4}$

x	-7	-6	-5	-4	-3
$f(x)$	0.149	0.406	1.104	3	8.155



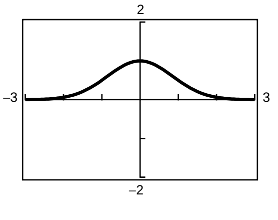
43. $f(x) = 2 + e^{x-5}$

x	2	3	4	5	6	7
$f(x)$	2.05	2.135	2.368	3	4.718	9.389



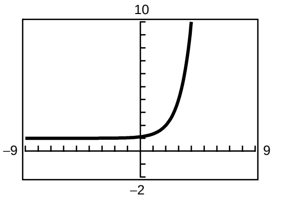
45. $y = 2^{-x^2}$

Asymptote: $y = 0$



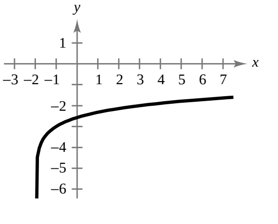
47. $f(x) = 3^{x-2} + 1$

Asymptote: $y = 1$

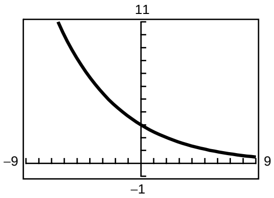


49. $y = 1.08^{-5x}$

Asymptote: $y = 0$



51. $S(t) = 3e^{-0.2t}$



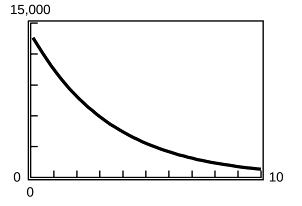
Asymptote: $S(t) = 0$

53. (a)

x	-1	-0.5	0	0.5	1
$f(x)$	$\frac{1}{3}$	0.577	1	1.732	3
$g(x)$	$\frac{1}{4}$	0.5	1	2	4

$4^x < 3^x$ when $x < 0$

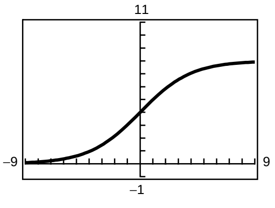
(b)



(i) $4^x < 3^x$ when $x > 0$

(ii) $4^x > 3$ when $x > 0$

55. (a)

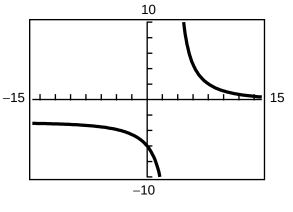


Horizontal asymptotes: $y = 0, y = 8$

(b)

x	-30	-20	-10	0	10	20	30
$f(x)$	≈ 0	≈ 0	0.05	4	7.95	≈ 8	≈ 8

57. (a)



(b)

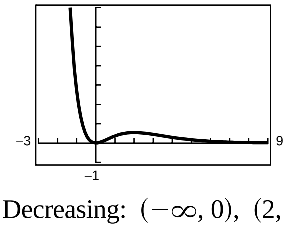
x	-20	-10	0	3	3.4	3.46	3.47	4	5	10	20
$f(x)$	-3.03	-3.22	-6	-34	-230	-2617	3516	26.6	8.4	1.11	0.11

Horizontal asymptotes: $y = -3, y = 0$

Veritcal asymptote: $x \approx 3.46$

59. $f(x) = x^2e^{-x}$

(a)



(b) Decreasing: $(-\infty, 0)$, $(2, \infty)$

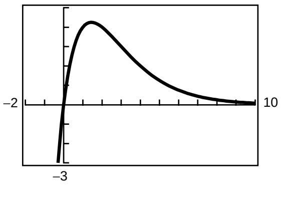
Increasing: $(0, 2)$

(c) Relative maximum: $(2, 4e^{-2}) \approx (2, 0.541)$

Relative minimum: $(0, 0)$

61. $f(x) = x2^{3-x}$

(a)



(b) Decreasing: $(1.44, \infty)$

Increasing: $(-\infty, 1.44)$

(c) Relative maximum: $(1.44, 4.25)$

63. $P = 2500, r = 8\% = 0.08, t = 10$

Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{.08}{n}\right)^{10n}$

Compounded continuously: $A = Pe^{rt} = 2500e^{(.08)(10)}$

n	1	2	4	12	365	Continuous
A	5397.31	5477.81	5520.10	5549.10	5563.36	5563.85

65. $P = 2500, r = 8\% = 0.08, t = 20$

Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{.08}{n}\right)^{20n}$

Compounded continuously: $A = Pe^{rt} = 2500e^{(.08)(20)}$

n	1	2	4	12	365	Continuous
A	11652.39	12002.55	12188.60	12317.01	12380.41	12382.58

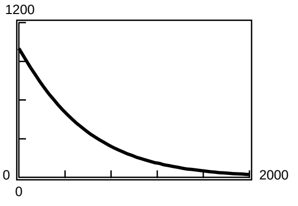
67. $P = 12{,}000, r = 8\% = 0.08$, compounded continuously: $A = Pe^{rt} = 12{,}000e^{(.08)t}$

t	1	10	20	30	40	50
A	12,999.44	26,706.49	59,436.39	132,278.12	294,390.36	655,177.80

69. $P = 12{,}000, r = 6.5\% = 0.065, A = P\left(1 + \frac{r}{n}\right)^{nt} = 12{,}000\left(1 + \frac{0.065}{12}\right)^{12t}$

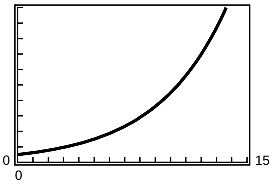
t	1	10	20	30	40	50
A	12,803.66	22,946.21	43,877.36	83,901.58	160,435.23	306,781.64

71. $P = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$

(a) 

(b) If $x = 500$, $p \approx \$421.12$

(c) For $x = 600$, $p \approx \$350.13$.

73. (a) 

(b) $P(0) = 100$
$P(5) \approx 300$
$P(10) \approx 900$

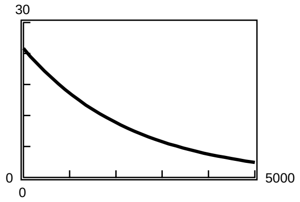
(c) $P(0) = 100e^{0.2197(0)} = 100$
$P(5) = 100e^{0.2197(5)} = 299.966 \approx 300$
$P(10) = 100e^{0.2197(10)} = 899.798 \approx 900$

75. $Q = 25\left(\frac{1}{2}\right)^{t/1620}$

(a) When $t = 0$, $Q = 25\left(\frac{1}{2}\right)^{0/1620} = 25(1) = 25$ grams.

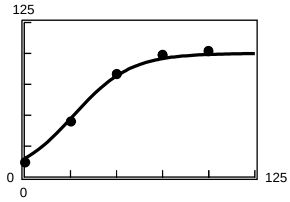
(b) When $t = 1000$, $Q = 25\left(\frac{1}{2}\right)^{1000/1620} \approx 16.30$ grams.

(c)



(d) No, $Q \to 0$ as $t \to \infty$, but Q never reaches 0.

77. (a) and (b)



The model fits the data well.

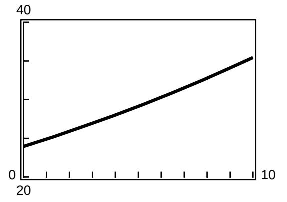
(c)

x	0	25	50	75	100
y	15	47	82	96	99

(d) If $x = 36$, $y \approx 64.7\%$.

(e) If $y = 66.7\%$, $x \approx 37.4$.

79. (a)

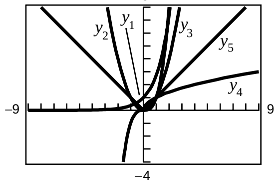


(b) $P(10) \approx 35.45$

(c) $P(10) = 23.95(1.04)^{10} \approx 35.45$

81. True. As $x \to -\infty$, $f(x) \to 0$

83.



(a) $y_1 = e^x$ increases at the fastest rate.

(b) For any positive interger n, $e^x > x^n$ for x sufficiently large. That is, e^x grows faster than x^n.

(c) A quantity is growing exponentially if its growth rate is of the form $y = ce^{rx}$. This is a faster rate than any polynomial growth rate.

85. Since $\sqrt{2} \approx 1.414$, we know that $1 < \sqrt{2} < 2$.

Thus: $2^1 < 2^{\sqrt{2}} < 2^2$

$2 < 2^{\sqrt{2}} < 4$