122. Given a real number r between -1 and 1, r^n approaches 0 as n increases. Thus the terms of a geometric sequence,

$a_1, a_1r, a_1r^2, a_1r^3, \ldots$

decrease in magnitude.

124. Let $x =$ amount of cement. Then $90 - x$ is the amount of sand. Thus,

$$\frac{x}{90 - x} = \frac{1}{4}$$
$$4x = 90 - x$$
$$5x = 90$$
$$x = 18 \text{ pounds of cement}$$
$$\Rightarrow 90 - 18 = 72 \text{ pounds of sand.}$$

126. Let $2n$ and $2n + 2$ be the two consecutive even integers.

$$(2n)(2n + 2) = 624$$
$$4n^2 + 4n - 624 = 0$$
$$n^2 + n - 156 = 0$$
$$(n - 12)(n + 13) = 0$$

Since the integers are positive, $n = 12$, and the two integers are 24 and 26.

128. $\begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 28 \end{bmatrix}$

130. $\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -11 & 17 & -1 \\ -30 & 24 & 32 \\ -22 & 13 & 36 \end{bmatrix}$

132. $\displaystyle\sum_{j=1}^{4} (3i + 4) = 7 + 10 + 13 + 16 = 46$

134. $\displaystyle\sum_{k=1}^{5} 12 = 5(12) = 60$

Section 9.4 Mathematical Induction

Solutions to Even-Numbered Exercises

2. $P_k = \dfrac{4}{(k + 2)(k + 3)}$

$$P_{k+1} = \frac{4}{[(k + 1) + 2][(k + 1) + 3]} = \frac{4}{(k + 3)(k + 4)}$$

4. $P_k = \dfrac{k}{2}(5x - 3)$

$$P_{k+1} = \frac{k + 1}{2}(5(k + 1) - 3) = \frac{k + 1}{2}(5k + 2)$$

6. $P_k = 7 + 13 + 19 + \cdots + [6(k - 1) + 1] + (6k + 1)$

$$P_{k+1} = 7 + 13 + 19 + \cdots + (6k + 1) + (6(k + 1) + 1)$$
$$= 7 + 13 + 19 + \cdots + (6k + 1) + (6k + 7)$$

8. 1. When $n = 1, S_1 = 1 = 1(2(1) - 1)$

2. Assume that

$$S_k = 1 + 5 + 9 + \cdots . + (4k - 3) = k(2k - 1)$$

Then

$$\begin{aligned} S_{k+1} &= 1 + 5 + 9 + \cdots + (4k - 3) + [4(k + 1) - 3] \\ &= 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 1) \\ &= Sk + (4k + 1) \\ &= k(2k - 1) + (4k + 1) \\ &= 2k^2 + 3k + 1 \\ &= (k + 1)(2k + 1) \\ &= (k + 1)(2(k + 1) - 1) \end{aligned}$$

We conclude by mathematical induction that the formula is valid for all positive integers n.

10. 1. When $n = 1$,

$$S_1 = 1 = \frac{1}{2}(3 \cdot 1 - 1).$$

2. Assume that $S_k = 1 + 4 + 7 + 10 + \cdots + (3k - 2) = \frac{k}{2}(3k - 1)$.

Then,

$$\begin{aligned} S_{k+1} &= 1 + 4 + 7 + 10 + \cdots + (3k - 2) + (3(k + 1) - 2) \\ &= S_k + (3(k + 1) - 2) \\ &= \frac{k}{2}(3k - 1) + (3k + 1) \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k + 1)(3k + 2)}{2} \\ &= \frac{k + 1}{2}[3(k + 1) - 1]. \end{aligned}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

12. 1. When $n = 1$, $S_1 = 2 = 3^1 - 1$.

2. Assume that

$$S_k = 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}) = 3^k - 1.$$

Then,

$$\begin{aligned} S_{k+1} &= 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}) + 2 \cdot 3^{k+1-1} \\ &= S_k + 2 \cdot 3^k \\ &= 3^k - 1 + 2 \cdot 3^k \\ &= 3 \cdot 3^k - 1 \\ &= 3^{k+1} - 1. \end{aligned}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

14. 1. When $n = 1$, $S_1 = 1^3 = 1 = \dfrac{1(1+1)^2}{4}$.

2. Assume that

$$S_k = 1^3 + 2^3 + 3^3 + 4^3 + \cdots . + k^3 = \frac{k^2(k+1)^2}{4}.$$

Then,

$$S_{k+1} = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3$$

$$= S_k + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \frac{(k+1)^2(k+2)^2}{4}.$$

Therefore, we conclude that this formula holds for all positive integer values of

16. 1. When $n = 1$, $S_1 = 2 = 1 + 1$.

2. Assume that

$$S_k = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{k}\right) = k + 1.$$

Then,

$$\begin{aligned} S_{k+1} &= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k+1}\right) \\ &= (S_k)\left(1 + \frac{1}{k+1}\right) \\ &= (k+1)\left(1 + \frac{1}{k+1}\right) \\ &= k + 1 + 1 \\ &= k + 2. \end{aligned}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

18. 1. When $n = 1$, $S_1 = \dfrac{(1)^2(1+1)^2(2(1)^2+2(1)-1)}{12} = 1.$

2. Assume that

$$S_k = \sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12}.$$

Then,

$$\begin{aligned}
S_{k+1} &= \sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} i^5 + (k+1)^5 \\
&= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + \frac{12(k+1)^5}{12} \\
&= \frac{(k+1)^2[k^2(2k^2+2k-1)+12(k+1)^3]}{12} \\
&= \frac{(k+1)^2[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)]}{12} \\
&= \frac{(k+1)^2[2k^4+14k^3+35k^2+36k+12]}{12} \\
&= \frac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12} \\
&= \frac{(k+1)^2(k+2)^2[2(k+1)^2+2(k+1)-1]}{12}.
\end{aligned}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

20. 1. When $n = 1$,

$$S_1 = \frac{1}{(1)(3)} = \frac{1}{2+1}$$

2. Assume that

$$S_k = \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}.$$

Then,

$$\begin{aligned}
S_{k+1} &= S_k + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2(k+1)+1}.
\end{aligned}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

22. 1. When $n = 7$, $\left(\frac{4}{3}\right)^7 \approx 7.4915 > 7$.

2. Assume that $\left(\frac{4}{3}\right)^k > k$, $k > 7$.

Then, $\left(\frac{4}{3}\right)^{k+1} = \left(\frac{4}{3}\right)^k \left(\frac{4}{3}\right) > k\left(\frac{4}{3}\right) = k + \frac{k}{3} > k + 1$ for $k > 7$.

Thus, $\left(\frac{4}{3}\right)^{k+1} > k + 1$.

Therefore, $\left(\frac{4}{3}\right)^n > n$.

24. 1. When $n = 1$, $\left(\frac{x}{y}\right)^2 < \left(\frac{x}{y}\right)$ and $(0 < x < y)$.

2. Assume that $\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k$

$$\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k \Longrightarrow \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^k \Longrightarrow \left(\frac{x}{y}\right)^{k+2} < \left(\frac{x}{y}\right)^{k+1}.$$

Therefore, $\left(\frac{x}{y}\right)^{n+1} < \left(\frac{x}{y}\right)^n$ for all integers $n \ge 1$.

26. 1. When $n = 1$, $3^1 > (1)2^1$

2. Assume that $3^k > k2^k$, $k \ge 2$.

First note that $k \ge 2 \Longrightarrow 3k \ge 2k + 2 = 2(k + 1)$

Then, $3^{k+1} = 3(3^k) > 3(k2^k) = (3k)2^k \ge 2(k + 1)2^k = (k + 1)2^{k+1}$.

Therefore $3^n > n2^n$ for all integers $n \ge 1$.

28. 1. When $n = 1$, $\left(\frac{a}{b}\right)^1 = \frac{a^1}{b^1}$.

2. Assume that $\left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$.

Then, $\left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k \left(\frac{a}{b}\right) = \frac{a^k}{b^k} \cdot \frac{a}{b} = \frac{a^{k+1}}{b^{k+1}}$.

Thus, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

30. 1. When $n = 1$, $\ln x_1 = \ln x_1$.

2. Assume that

$$\ln(x_1 x_2 x_3 \ldots x_k) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_k .$$

Then,
$$\begin{aligned}\ln(x_1 x_2 x_3 \ldots x_k\ x_{k+1}) &= \ln[(x_1 x_2 x_3 \ldots x_k) x_{k+1}] \\ &= \ln(x_1 x_2 x_3 \ldots x_k) + \ln x_{k+1} \\ &= \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_k + \ln x_{k+1}.\end{aligned}$$

Thus, $\ln(x_1 x_2 x_3 \ldots x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_n$.

32. 1. When $n = 1$, $a + bi$ and $a - bi$ are complex conjugates by definition.

2. Assume that $(a + bi)^k$ and $(a - bi)^k$ are complex conjugates.

That is, if $(a + bi)^k = c + di$, then $(a - bi)^k = c - di$.

Then,

$$\begin{aligned}(a + bi)^{k+1} &= (a + bi)^k(a + bi) = (c + di)(a + bi)\\ &= (ac - bd) + i(bc + ad)\\ \text{and } (a - bi)^{k+1} &= (a - bi)^k(a - bi) = (c - di)(a - bi)\\ &= (ac - bd) - i(bc + ad).\end{aligned}$$

This implies that $(a + bi)^{k+1}$ and $(a - bi)^{k+1}$ are complex conjugates. Therefore, $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for $n \geq 1$.

34. 1. When $n = 1$, $(2^{2(1)-1} + 3^{2(1)-1}) = 2 + 3 = 5$ and 5 is a factor.

2. Assume that 5 is a factor of $(2^{2k-1} + 3^{2k-1})$.

Then,
$$\begin{aligned}(2^{2(k+1)-1} + 3^{2(k+1)-1}) &= (2^{2k+2-1} + 3^{2k+2-1})\\ &= (2^{2k-1}2^2 + 3^{2k-1}3^2)\\ &= (4 \cdot 2^{2k-1} + 9 \cdot 3^{2k-1})\\ &= (2^{2k-1} + 3^{2k-1}) + (2^{2k-1} + 3^{2k-1})\\ &\quad + (2^{2k-1} + 3^{2k-1}) + (2^{2k-1} + 3^{2k-1}) + 5 \cdot 3^{2k-1}.\end{aligned}$$

Since 5 is a factor of each set of parenthesis and 5 is a factor of $5 \cdot 3^{2k-1}$, then 5 is a factor of the whole sum. Thus, 5 is a factor of $(2^{2n-1} + 3^{2n-1})$ for every positive integer n.

36. $a_0 = 1, a_n = a_{n-1} + 2$

$a_0 = 1$

$a_1 = a_0 + 2 = 1 + 2 = 3$

$a_2 = a_1 + 2 = 3 + 2 = 5$

$a_3 = a_2 + 2 = 5 + 2 = 7$

$a_4 = a_3 + 2 = 7 + 2 = 9$

38. $a_0 = 4, a_1 = 2, a_n = a_{n-1} - a_{n-2}$

$a_0 = 4$

$a_1 = 2$

$a_2 = a_1 - a_0 = 2 - 4 = -2$

$a_3 = a_2 - a_1 = -2 - 2 = -4$

$a_4 = a_3 - a_2 = -4(-2) = -2$

40. $a_1 = 0, a_n = a_{n-1} + 3$

$a_1 = 0$

$a_2 = a_1 + 3 = 0 + 3 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 3 = 6 + 3 = 9$

$a_5 = a_4 + 3 = 9 + 3 = 12$

a_n : 0 3 6 9 12

First differences : 3 3 3 3

Second differences : 0 0 0

Since the first differences are equal, the sequence has a linear model.

42. $a_1 = 3, a_n = a_{n-1} - n$

$a_1 = 3$

$a_2 = a_1 - 2 = 3 - 2 = 1$

$a_3 = a_2 - 3 = 1 - 3 = -2$

$a_4 = a_3 - 4 = -2 - 4 = -6$

$a_5 = a_4 - 5 = -6 - 5 = -11$

a_n : 3 1 −2 −6 −11

First differences : −2 −3 −4 −5

Second differences : −1 −1 −1

Since the second differences are all the same, the sequence has a quadratic model.

44. $a_0 = 0, a_n = a_{n-1} + n$

$a_0 = 0$

$a_1 = a_0 + 1 = 0 + 1 = 1$

$a_2 = a_1 + 2 = 1 + 2 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 4 = 6 + 4 = 10$

a_n :	0		1		3		6		10
First differences :		1		2		3		4	
Second differences :			1		1		1		

Since the second differences are equal, the sequence has a quadratic model.

46. $a_1 = 2, a_n = a_{n-1} + 2$

$a_1 = 2$

$a_2 = a_1 + 2 = 2 + 2 = 4$

$a_3 = a_2 + 2 = 4 + 2 = 6$

$a_4 = a_3 + 2 = 6 + 2 = 8$

$a_5 = a_4 + 2 = 8 + 2 = 10$

a_n :	2		4		6		8		10
First differences :		2		2		2		2	
Second differences :			0		0		0		

Since the first differences are equal, the sequence has a linear model.

48. $a_0 = 1, a_n = a_{n-1} + n^2$

$a_0 = 1$

$a_1 = 1 + 1^2 = 2$

$a_2 = 2 + 2^2 = 6$

$a_3 = 6 + 3^2 = 15$

$a_4 = 15 + 4^2 = 31$

a_n :	1		2		6		15		31
First differences :		1		4		9		16	
Second differences :			3		5		7		

Since neither the first differences, nor the second differences are equal, the sequence does not have a linear or a quadratic model.

50. $a_0 = 3, a_1 = 3, a_4 = 15$

Let $a_n = an^2 + bn + c$. Thus

$a_0 = a(0)^2 + b(0) + c = 3 \implies c = 3$

$a_1 = a(1)^2 + b(1) + c = 3 \implies a + b + c = 3$

$a + b = 0$

$a_4 = a(4)^2 + b(4) + c = 15 \implies 16a + 4b + c = 15$

$16a + 4b = 12$

$4a + b = 3$

By elimination: $-a - b = 0$

$4a + b = 3$

$3a = 3$

$a = 1 \implies b = -1$

Thus, $a_n = n^2 - n + 3$.

52. $a_0 = -3, a_2 = 1, a_4 = 9$

Let $a_n = an^2 + bn + c$. Then

$$\begin{aligned}
a_0 &= a(0)^2 + b(0) + c = -3 &&\Longrightarrow\ c = -3\\
a_2 &= a(2)^2 + b(2) + c = 1 &&\Longrightarrow\ 4a + 2b + c = 1\\
&&& \quad\ 4a + 2b = 4\\
&&& \quad\ 2a + b = 2\\
a_4 &= a(4)^2 + b(4) + c = 9 &&\Longrightarrow\ 16a + 4b + c = 9\\
&&& \quad\ 16a + 4b = 12\\
&&& \quad\ 4a + b = 3
\end{aligned}$$

By elimination:
$$\begin{aligned}
-2a - b &= -2\\
4a + b &= 3\\
\hline
2a &= -1\\
a &= \tfrac{1}{2} \Longrightarrow b = 1
\end{aligned}$$

Thus, $a_n = \frac{1}{2}n^2 + n - 3$.

54. False. P_1 might not even be defined.

56. False. It has $n - 2$ second differences.

58. (a) If P_3 is true and P_k implies P_{k+1}, then P_n is true for integers $n \geq 3$.

(b) If $P_1, P_2, P_3, \ldots, P_{50}$ are all true, then P_n is true for integers $1 \leq n \leq 50$.

(c) If P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true, then you may only conclude that P_1, P_2, and P_3 are true.

(d) If P_2 is true and P_{2k} implies P_{2k+2}, then P_{2n} is true for any positive integer n.

60. $\begin{bmatrix} 1 & -3 & : & 1 \\ 7 & -6 & : & -38 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & : & -8 \\ 0 & 1 & : & -3 \end{bmatrix}$

Answer: $(-8, -3)$

62. $x - y^3 = 0 \Longrightarrow x = y^3$

$x - 2y^2 = 0$

$y^3 - 2y^2 = 0$

$y^2(y - 2) = 0 \Longrightarrow y = 0, 2$

When $y = 0$: $x = 0^3 = 0$.

When $y = 2$: $x = 2^3 = 8$.

Points of intersection: $(0, 0)$ and $(8, 2)$

64.
$$\begin{aligned}
2x + y - 2z &= 1\\
x - z &= 1\\
3x + 3y + z &= 12
\end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 3 & 3 & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{4}\begin{bmatrix} -3 & 7 & 1 \\ 4 & -8 & 0 \\ -3 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 & 7 & 1 \\ 4 & -8 & 0 \\ -3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

Thus, $x = 4, y = -1, z = 3$.

Answer: $(4, -1, 3)$