Section 10.5 Parametric Equations

- If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a *plane curve C*. The equations $x = f(t)$ and $y = g(t)$ are *parametric equations* for C and t is the *parameter.*
- You should be able to graph plane curves with your graphing utility.
- To eliminate the parameter:

  Solve for t in one equation and substitute into the second equation.
- You should be able to find the parametric equations for a graph.

Solutions to Odd-Numbered Exercises

1. $x = t$

$y = t + 2$

$y = x + 2$ line

Matches (c).

3. $x = \sqrt{t}$

$y = t$

$y = x^2$ parabola, $x \geq 0$

Matches (b).

5. $x = \dfrac{1}{t} \implies t = \dfrac{1}{x}$

$y = t + 2$

$y = \dfrac{1}{x} + 2$

Matches (a).

7. $x = \ln t \iff t = e^x$

$y = \dfrac{1}{2}t - 2$

$y = \dfrac{1}{2}e^x - 2$

Matches (f).

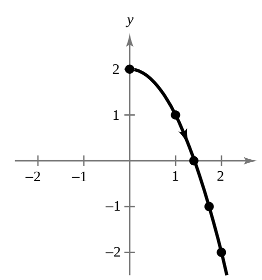
9. $x = \sqrt{t}, y = 2 - t$

(a)

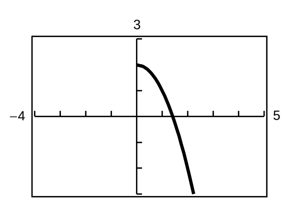
t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	2	1	0	-1	-2

(b) Graph by hand

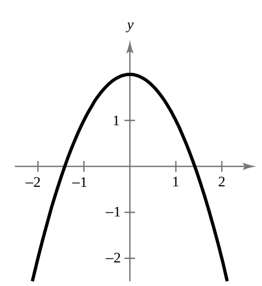
Note: $x \geq 0$



(c)



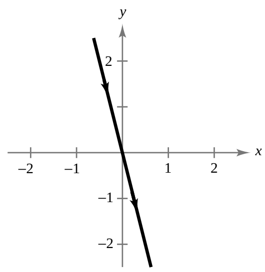
(d) $y = 2 - t = 2 - x^2$ parabola



In part (c), $x \geq 0$.

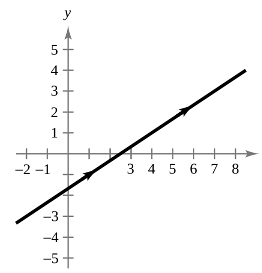
11. $x = t, y = -4t$

$y = -4x$



13. $x = 3t + 1, y = 2t - 1$

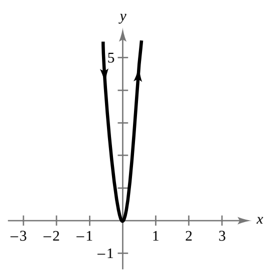
$y = 2\left(\frac{x-1}{3}\right) - 1 = \frac{2}{3}x - \frac{5}{3}$ or $2x - 3y - 5 = 0$



15. $x = \frac{1}{4}t,\ y = t^2$

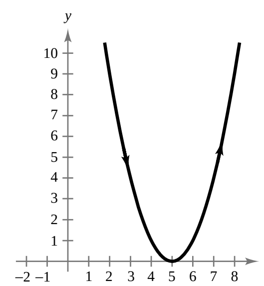
$y = (4x)^2$

$y = 16x^2$



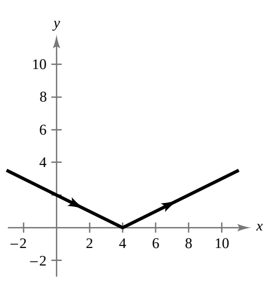
17. $x = t + 5, y = t^2$

$y = (x - 5)^2$



19. $x = 2t$

$y = |t - 2|$



$$t = \frac{x}{2} \implies y = |t - 2|$$

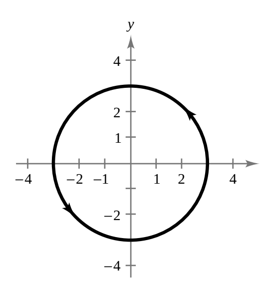
$$= \left|\frac{x}{2} - 2\right|$$

$$= \frac{1}{2}|x - 4|$$

21. $x = 3\cos\theta \implies \left(\frac{x}{3}\right)^2 = \cos^2\theta$

$y = 3\sin\theta \implies \left(\frac{y}{3}\right)^2 = \sin^2\theta$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \implies x^2 + y^2 = 9$

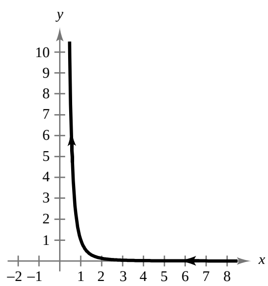


23. $x = e^{-t} \implies \dfrac{1}{x} = e^{t}$

$y = e^{3t} \implies y = (e^{t})^3$

$y = \left(\dfrac{1}{x}\right)^3$

$y = \dfrac{1}{x^3},\ x > 0,\ y > 0$

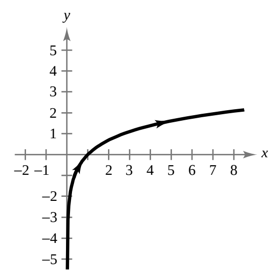


25. $x = t^3 \implies x^{1/3} = t$

$y = 3 \ln t \implies y = \ln t^3$

$y = \ln(x^{1/3})^3$

$y = \ln x$



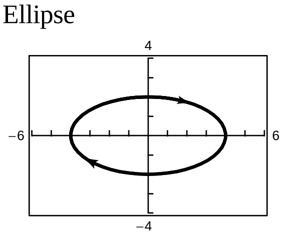
27. $x = 4 \sin 2\theta \implies \left(\dfrac{x}{4}\right)^2 = \sin^2 2\theta$

$y = 2 \cos 2\theta \implies \left(\dfrac{y}{2}\right)^2 = \cos^2 2\theta$

$\dfrac{x^2}{16} + \dfrac{y^2}{4} = \sin^2 2\theta + \cos^2 2\theta$

$\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

Ellipse



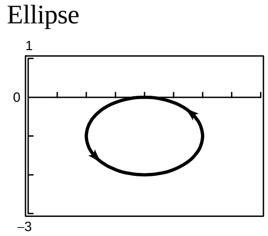
29. $x = 4 + 2\cos\theta \implies \left(\dfrac{x-4}{2}\right)^2 = \cos^2\theta$

$y = -1 + \sin\theta \qquad (y+1)^2 = \sin^2\theta$

$\left(\dfrac{x-4}{2}\right)^2 + (y+1)^2 = \cos^2\theta + \sin^2\theta$

$\dfrac{(x-4)^2}{4} + (y+1)^2 = 1$

Ellipse



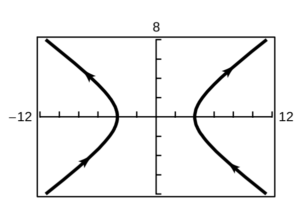
31. $x = 4 \sec\theta \implies \left(\dfrac{x}{4}\right)^2 = \sec^2\theta$

$y = 3\tan\theta \implies \left(\dfrac{y}{3}\right)^2 = \tan^2\theta$

$\left(\dfrac{x}{4}\right)^2 - \left(\dfrac{y}{3}\right)^2 = \sec^2\theta - \tan^2\theta$

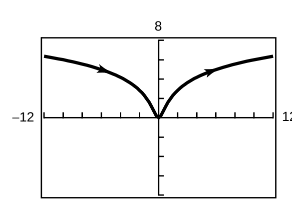
$\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

Hyperbola



33. $x = \dfrac{t}{2}$

$y = \ln(t^2 + 1)$



35. By eliminating the parameters in (a)–(d), we get $y = 2x + 1$. They differ from each other in restricted domain and in orientation.

(a) Domain: $-\infty < x < \infty$
Orientation: Left to right

(b) Domain: $-1 \le x \le 1$
Orientation: Depends on θ

(c) Domain: $0 < x < \infty$
Orientation: Right to left

(d) Domain: $0 < x < \infty$
Orientation: Left to right

37. $t = \dfrac{(x - x_1)}{(x_2 - x_1)}$

$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$

$$\Rightarrow y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

39. $x = h + a\cos\theta$

$y = k + b\sin\theta$

$$\frac{x - h}{a} = \cos\theta, \frac{y - k}{b} = \sin\theta$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

41. $x = x_1 + t(x_2 - x_1) = 0 + t(5 - 0) = 5t$

$y = y_1 + t(y_2 - y_1) = 0 + t(-2 - 0) = -2t$

(Solution not unique.)

43. From Exercise 38:

$x = 2 + 4\cos\theta$
$y = 1 + 4\sin\theta$
Solution not unique

45. From Exercise 39:

$a = 5$, $c = 4$, and hence, $b = 3$.
$x = 5\cos\theta$
$y = 3\sin\theta$
Center: $(0, 0)$
Solution not unique

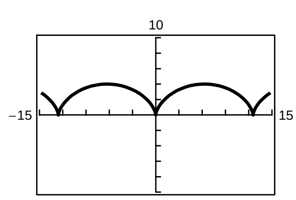
47. $y = 3x - 2$

Examples:

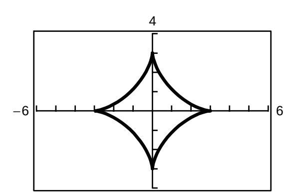
$x = t$ $\qquad$ $x = 2t$
$y = 3t - 2$ $\qquad$ $y = 6t - 2$

49. $x = 2(\theta - \sin\theta)$

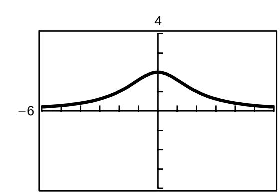
$y = 2(1 - \cos\theta)$



51. $x = 3\cos^3\theta$, $y = 3\sin^3\theta$



53. $x = 2\cot\theta$, $y = 2\sin^2\theta$



55. Matches graph (b).

57. Matches graph (d).

59. $x = (v_0 \cos \theta)t, \; y = h + (v_0 \sin \theta)t - 16t^2$

(a) $100 \text{ miles/hour} = \dfrac{100 \text{ mi/hr} \cdot 5280 \text{ ft/mi}}{3600 \text{ sec/hr}}$

$= 146.67 \text{ ft/sec}$

$x = (146.67 \cos \theta)t$

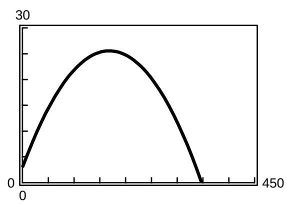
$y = 3 + (146.67 \sin \theta)t - 16t^2$

(b) $\theta = 15°$

$x = (146.67 \cos 15°)t = 141.7t$

$y = 3 + (146.67 \sin 15°)t - 16t^2$

$= 3 + 38.0t - 16t^2$



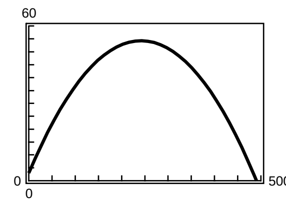
It is not a home run because $y < 10$ when $x = 400$.

(c) $\theta = 23°$

$x = (146.67 \cos 23°)t = 135.0t$

$y = 3 + (146.67 \sin 23°)t - 16t^2$

$= 3 + 57.3t - 16t^2$



Yes, it is a home run because $y > 10$ when $x = 400$.

(d) $\theta = 19.4°$ is the minimum angle.

61. True

$x = t$ first set

$y = t^2 + 1 = x^2 + 1$

$x = 3t$ second set

$y = 9t^2 + 1 = (3t)^2 + 1 = x^2 + 1$

63. The graph is the same, but the orientation is reversed.

65. $x^2 - 6x + 4 = 0$

$$x = \frac{6 \pm \sqrt{36 - 16}}{2} = 3 \pm \sqrt{5}$$

67. $x^4 - 18x^2 + 18 = 0$

$$x^2 = \frac{18 \pm \sqrt{18^2 - 4(18)}}{2} = 9 \pm \sqrt{63}$$

$$x = \pm\sqrt{9 + \sqrt{63}}, \pm\sqrt{9 - \sqrt{63}}$$

[Four solutions: $x \approx \pm 4.1155, \pm 1.0309$]

69. $\displaystyle\sum_{n=1}^{200} (n - 8) = \frac{200(201)}{2} - 8(200) = 18{,}500$

71. $\displaystyle\sum_{n=1}^{70} \frac{7 - 5n}{12} = \frac{7}{12}(70) - \frac{5}{12}\left[\frac{70(71)}{2}\right] = \frac{-11935}{12}$

≈ -994.5833

73. $\displaystyle\sum_{n=0}^{10} 10\left(\frac{2}{3}\right)^n = 10\frac{1 - \left(\frac{2}{3}\right)^{11}}{1 - \frac{2}{3}} = 30\left[1 - \left(\frac{2}{3}\right)^{11}\right] \approx 29.6532$