Section 10.6 Polar Coordinates

- In polar coordinates you do not have unique representation of points. The point (r, θ) can be represented by $(r, \theta \pm 2n\pi)$ or by $(-r, \theta \pm (2n + 1)\pi)$ where n is any integer. The pole is represented by $(0, \theta)$ where θ is any angle.
- To convert from polar coordinates to rectangular coordinates, use the following relationships.

  $x = r \cos \theta$

  $y = r \sin \theta$
- To convert from rectangular coordinates to polar coordinates, use the following relationships.

  $r = \pm\sqrt{x^2 + y^2}$

  $\tan \theta = y/x$

  If θ is in the same quadrant as the point (x, y), then r is positive. If θ is in the opposite quadrant as the point (x, y), then r is negative.
- You should be able to convert rectangular equations to polar form and vice versa.

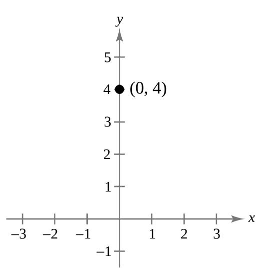
Solutions to Odd-Numbered Exercises

1. Polar coordinates: $\left(4, \frac{\pi}{2}\right)$

$x = 4 \cos\left(\frac{\pi}{2}\right) = 0$

$y = 4 \sin\left(\frac{\pi}{2}\right) = 4$

Rectangular coordinates: $(0, 4)$

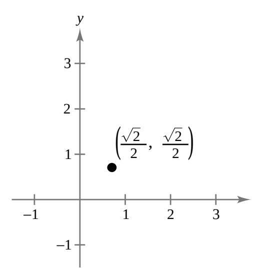


3. Polar coordinates: $\left(-1, \frac{5\pi}{4}\right)$

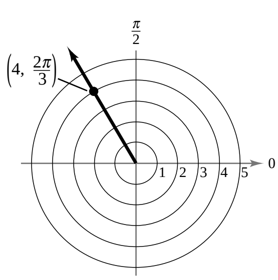
$x = -1 \cos\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$y = -1 \sin\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Rectangular coordinates: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



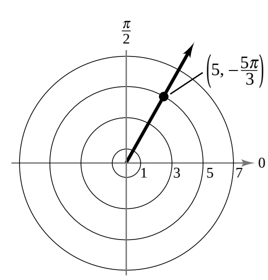
5.



Three additional representations:

$\left(-4, \frac{5\pi}{3}\right), \left(4, -\frac{4\pi}{3}\right), \left(-4, -\frac{\pi}{3}\right)$

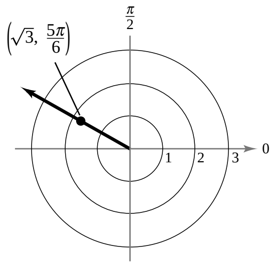
7.



Three additional representations:

$\left(5, \frac{\pi}{3}\right), \left(-5, \frac{4\pi}{3}\right), \left(-5, -\frac{2\pi}{3}\right)$

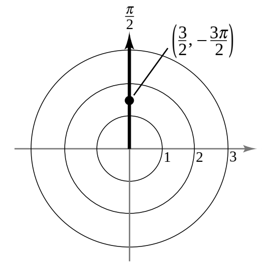
9.



Three additional representations:

$$\left(\sqrt{3}, -\frac{7\pi}{6}\right), \left(-\sqrt{3}, -\frac{\pi}{6}\right), \left(-\sqrt{3}, \frac{11\pi}{6}\right)$$

11.



Three additional representations:

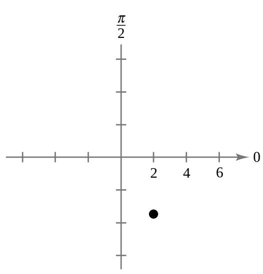
$$\left(\frac{3}{2}, \frac{\pi}{2}\right), \left(-\frac{3}{2}, \frac{3\pi}{2}\right), \left(-\frac{3}{2}, -\frac{\pi}{2}\right)$$

13. Polar coordinates: $\left(4, -\frac{\pi}{3}\right)$

$$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$$

$$y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$$

Rectangular coordinates: $\left(2, -2\sqrt{3}\right)$

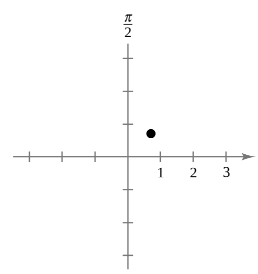


15. Polar coordinates: $\left(-1, \frac{-3\pi}{4}\right)$

$$x = -1\cos\left(\frac{-3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$y = -1\sin\left(\frac{-3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Rectangular coordinates: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

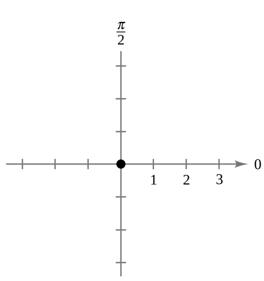


17. Polar coordinates: $\left(0, -\frac{7\pi}{6}\right)$ (origin!)

$$x = 0\cos\left(-\frac{7\pi}{6}\right) = 0$$

$$y = 0\sin\left(-\frac{7\pi}{6}\right) = 0$$

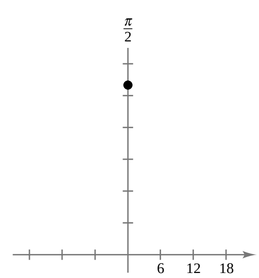
Rectangular coordinates: (0, 0)



19. Polar coordinates: $\left(32, \frac{5\pi}{2}\right)$

$$x = 32\cos\left(\frac{5\pi}{2}\right) = 0,\ y = 32\sin\left(\frac{5\pi}{2}\right) = 32$$

Rectangular coordinates: (0, 32)

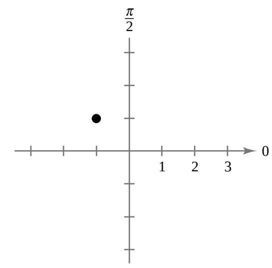


21. Polar coordinates: $(\sqrt{2}, 2.36)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

$y = \sqrt{2}\sin(2.36) \approx 0.996$

Rectangular coordinates: $(-1.004, 0.996)$



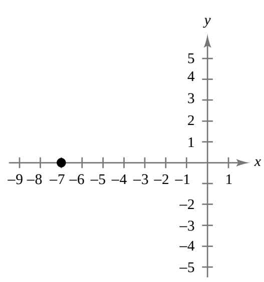
23. $(r, \theta) = \left(2, \frac{3\pi}{4}\right) \implies (x, y) = (-1.414, 1.414) = (-\sqrt{2}, \sqrt{2})$

25. $(r, \theta) = (-4.5, 1.3) \implies (x, y) = (-1.204, -4.336)$

27. Rectangular coordinates: $(-7, 0)$

$r = 7, \ \tan\theta = 0, \ \theta = 0$

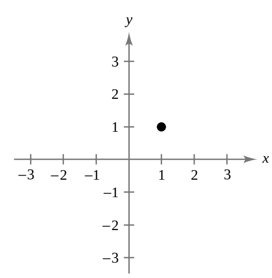
Polar coordinates: $(7, \pi), \ (-7, 0)$



29. Rectangular coordinates: $(1, 1)$

$r = \sqrt{2}, \ \tan\theta = 1, \ \theta = \frac{\pi}{4}$

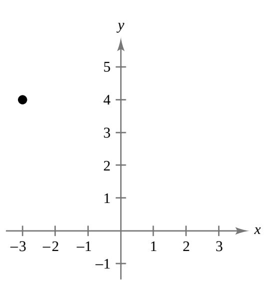
Polar coordinates: $\left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$



31. Rectangular coordinates: $(-3, 4)$

$r = \sqrt{9 + 16} = 5, \ \tan\theta = -\frac{4}{3}, \ \theta \approx 2.214$

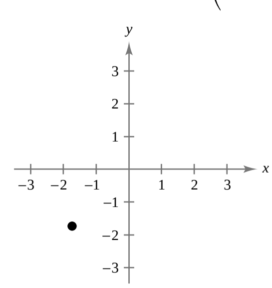
Polar coordinates: $(5, 2.214), \ (-5, 5.356)$



33. Rectangular coordinates: $(-\sqrt{3}, -\sqrt{3})$

$r = \sqrt{3 + 3} = \sqrt{6}, \ \tan\theta = 1, \ \theta = \frac{\pi}{4}$

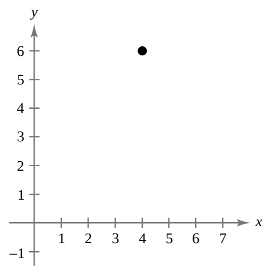
Polar coordinates: $\left(\sqrt{6}, \frac{5\pi}{4}\right), \left(-\sqrt{6}, \frac{\pi}{4}\right)$



35. Rectangular coordinates: (4, 6)

$r = \sqrt{16 + 36} = 2\sqrt{13}$, $\tan\theta = \dfrac{3}{2}$, $\theta \approx 0.983$

Polar coordinates: $(2\sqrt{13}, 0.983)$, $(-2\sqrt{13}, 4.124)$



37. $(x, y) = (3, -2) \implies r = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$\theta = \arctan\left(-\frac{2}{3}\right) \approx -0.588$

$(r, \theta) \approx (\sqrt{13}, -0.588)$

39. $(x, y) = (\sqrt{3}, 2) \implies r = \sqrt{3 + 2^2} = \sqrt{7}$

$\theta = \arctan\left(\dfrac{2}{\sqrt{3}}\right) \approx 0.857$

$(r, \theta) \approx (\sqrt{7}, 0.857)$

41. $(x, y) = \left(\dfrac{5}{2}, \dfrac{4}{3}\right) \implies r = \sqrt{\left(\dfrac{5}{2}\right)^2 + \left(\dfrac{4}{3}\right)^2} = \dfrac{17}{6}$

$\theta = \arctan\left(\dfrac{4/3}{5/2}\right) \approx 0.490$

$(r, \theta) \approx \left(\dfrac{17}{6}, 0.490\right)$

43. $(x, y) = (0, -5) \implies (r, \theta) = (5, -1.571) = \left(5, -\dfrac{\pi}{2}\right)$

45. (a) $x^2 + y^2 = 49$

$r^2 = 49$

$r = 7$

(b) $x^2 + y^2 = a^2 \quad (a > 0)$

$r = a$

47. (a) $x^2 + y^2 - 2ax = 0$

$r^2 - 2a\cos\theta = 0$

$r(r - 2a\cos\theta) = 0$

$r = 2a\cos\theta$

(b) $x^2 + y^2 - 2ay = 0$

$r^2 - 2a\sin\theta = 0$

$r(r - 2a\sin\theta) = 0$

$r = 2a\sin\theta$

49. (a) $x = 12$

$r\cos\theta = 12$

$r = 12\sec\theta$

(b) $x = a$

$r\cos\theta = a$

$r = a\sec\theta$

51. (a) $xy = 4$

$(r\cos\theta)(r\sin\theta) = 4$

$r^2\cos\theta\sin\theta = 4$

$r^2(2\cos\theta\sin\theta) = 8$

$r^2\sin 2\theta = 8$

$r^2 = 8\csc 2\theta$

(b) $2xy = 1$

$2(r\cos\theta)(r\sin\theta) = 1$

$r^2(2\cos\theta\sin\theta) = 1$

$r^2\sin 2\theta = 1$

$r^2 = \csc 2\theta$

53. (a) $y^2 = x^3$

$(r\sin\theta)^2 = (r\cos\theta)^3$

$\sin^2\theta = r\cos^3\theta$

$r = \dfrac{\sin^2\theta}{\cos^3\theta} = \tan^2\theta\sec\theta$

(b) $x^2 = y^3$

$(r\cos\theta)^2 = (r\sin\theta)^3$

$\cos^2\theta = r\sin^3\theta$

$r = \dfrac{\cos^2\theta}{\sin^3\theta} = \cot^2\theta\csc\theta$

55.
$$\begin{aligned} r &= 4\sin\theta \\ r^2 &= 4r\sin\theta \\ x^2 + y^2 &= 4y \\ x^2 + y^2 - 4y &= 0 \end{aligned}$$

57.
$$\begin{aligned} \theta &= \frac{\pi}{6} \\ \tan\theta &= \frac{\sqrt{3}}{3} \\ \frac{y}{x} &= \frac{\sqrt{3}}{3} \\ y &= \frac{\sqrt{3}}{3}x \\ \sqrt{3}x - 3y &= 0 \end{aligned}$$

59.
$$\begin{aligned} r &= 4 \\ r^2 &= 16 \\ x^2 + y^2 &= 16 \end{aligned}$$

61.
$$\begin{aligned} r &= -3\csc\theta \\ r\sin\theta &= -3 \\ y &= -3 \end{aligned}$$

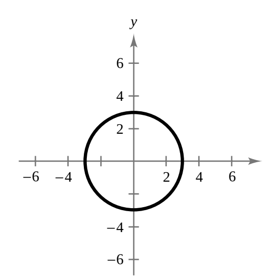
63.
$$\begin{aligned} r^2 &= \cos\theta \\ r^3 &= r\cos\theta \\ (x^2 + y^2)^{3/2} &= x \\ x^2 + y^2 &= x^{2/3} \\ (x^2 + y^2)^3 &= x^2 \end{aligned}$$

65.
$$\begin{aligned} r &= 2\sin 3\theta \\ r &= 2(3\sin\theta - 4\sin^3\theta) \\ r^4 &= 6r^3\sin\theta - 8r^3\sin^3\theta \\ (x^2 + y^2)^2 &= 6(x^2 + y^2)y - 8y^3 \\ (x^2 + y^2)^2 &= 6x^2y - 2y^3 \end{aligned}$$

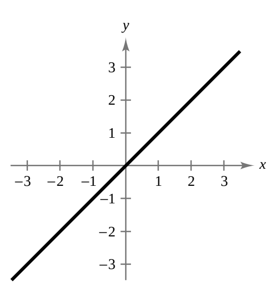
67.
$$\begin{aligned} r &= \frac{1}{1 - \cos\theta} \\ r - r\cos\theta &= 1 \\ \sqrt{x^2 + y^2} - x &= 1 \\ x^2 + y^2 &= 1 + 2x + x^2 \\ y^2 &= 2x + 1 \end{aligned}$$

69.
$$\begin{aligned} r &= \frac{6}{2 - 3\sin\theta} \\ r(2 - 3\sin\theta) &= 6 \\ 2r &= 6 + 3r\sin\theta \\ 2\left(\pm\sqrt{x^2 + y^2}\right) &= 6 + 3y \\ 4(x^2 + y^2) &= (6 + 3y)^2 \\ 4x^2 + 4y^2 &= 36 + 36y + 9y^2 \\ 4x^2 - 5y^2 - 36y - 36 &= 0 \end{aligned}$$

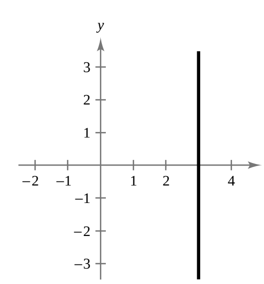
71.
$$\begin{aligned} r &= 3 \\ r^2 &= 9 \\ x^2 + y^2 &= 9 \end{aligned}$$



73.
$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan\theta &= \tan\frac{\pi}{4} \\ \frac{y}{x} &= 1 \\ y &= x \\ x - y &= 0 \end{aligned}$$



75.
$$\begin{aligned} r &= 3\sec\theta \\ r\cos\theta &= 3 \\ x &= 3 \\ x - 3 &= 0 \end{aligned}$$



77. True, the distances from the origin are the same.

79. (a) $(r_1, \theta_1) = (x_1, y_1)$ where $x_1 = r_1 \cos\theta_1$ and $y_1 = r_1 \sin\theta_1$.

$(r_2, \theta_2) = (x_2, y_2)$ where $x_2 = r_2 \cos\theta_2$ and $y_2 = r_2 \sin\theta_2$.

Then $x_1^2 + y_1^2 = r_1^2 \cos^2\theta_1 + r_1^2 \sin^2\theta_1 = r_1^2$ and $x_2^2 + y_2^2 = r_2^2$. Thus,

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2} \\ &= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)} \\ &= \sqrt{r_1^2 + r_2^2 - 2(r_1r_2\cos\theta_1\cos\theta_2 + r_1r_2\sin\theta_1\sin\theta_2)} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}. \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points are on the same line through the origin. In this case,

$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(0)} = \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|$.

(c) If $\theta_1 - \theta_2 = 90°$, $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem.

(d) For instance, $\left(3, \frac{\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$ gives $d \approx 2.053$ and $\left(-3, \frac{7\pi}{6}\right), \left(-4, \frac{4\pi}{3}\right)$ gives $d \approx 2.053$. (same!)

81. $D = \begin{vmatrix} 5 & -7 \\ -3 & 1 \end{vmatrix} = 5 - 21 = -16$

$D_x = \begin{vmatrix} -11 & -7 \\ -3 & 1 \end{vmatrix} = -11 - 21 = -32$

$D_y = \begin{vmatrix} 5 & -11 \\ -3 & -3 \end{vmatrix} = -15 - 33 = -48$

$x = \frac{D_x}{D} = \frac{-32}{-16} = 2$

$y = \frac{Dy}{D} = \frac{-48}{-16} = 3$

83. $D = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & -3 & 9 \end{vmatrix} = 35$

$D_x = \begin{vmatrix} 0 & -2 & 1 \\ 0 & 1 & -3 \\ 8 & -3 & 9 \end{vmatrix} = 40$

$D_y = \begin{vmatrix} 3 & 0 & 1 \\ 2 & 0 & -3 \\ 1 & 8 & 9 \end{vmatrix} = 88$

$D_z = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 8 \end{vmatrix} = 56$

$x = \frac{D_x}{D} = \frac{40}{35} = \frac{8}{7}$

$y = \frac{D_y}{D} = \frac{88}{35}$

$z = \frac{D_z}{D} = \frac{56}{35} = \frac{8}{5}$

85. $(x + 5)^8$. $ax^3 = 175{,}000x^3$. $a = 175{,}000$

87. $(2x - y)^{12}$. $ax^7y^5 = -101{,}376x^7y^5$. $a = -101{,}376$