Section 10.7 Graphs of Polar Equations

- When graphing polar equations:
  1. Test for symmetry
     (a) $\theta = \pi/2$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
     (b) Polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
     (c) Pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.
     (d) $r = f(\sin \theta)$ is symmetric with respect to the line $\theta = \pi/2$.
     (e) $r = f(\cos \theta)$ is symmetric with respect to the polar axis.
  2. Find the θ values for which $|r|$ is maximum.
  3. Find the θ values for which $r = 0$.
  4. Know the different types of polar graphs.
     (a) Limaçons
     $r = a \pm b \cos \theta$
     $r = a \pm b \sin \theta$
     (b) Rose Curves, $n \geq 2$
     $r = a \cos n\theta$
     $r = a \sin n\theta$
     (c) Circles
     $r = a \cos \theta$
     $r = a \sin \theta$
     $r = a$
     (d) Lemniscates
     $r^2 = a^2 \cos 2\theta$
     $r^2 = a^2 \sin 2\theta$
- You should be able to graph polar equations of the form $r = f(\theta)$ with your graphing utility. If your utility does not have a polar mode, use
  $x = f(t) \cos t$
  $y = f(t) \sin t$
  in parametric mode.

Solutions to Odd-Numbered Exercises

1. $r = 3 \cos 2\theta$ is a rose curve.

3. $r = 3 \cos \theta$ is a circle.

5. $r = 6 \sin 2\theta$ is a rose curve.

7. $r = 10 + 4 \cos \theta$

$\theta = \dfrac{\pi}{2}$: $-r = 10 + 4 \cos(-\theta)$

$-r = 10 + 4 \cos \theta$ Not an equivalent equation

$r = 10 + 4 \cos(\pi - \theta)$

$r = 10 + 4(\cos \pi \cos \theta + \sin \pi \sin \theta)$

$r = 10 - 4 \cos \theta$ Not an equivalent equation

Polar axis: $r = 10 + 4 \cos(-\theta)$

$r = 10 + 4 \cos \theta$ Equivalent equation

Pole: $-r = 10 + 4 \cos \theta$ Not an equivalent equation

$r = 10 + 4 \cos(\pi + \theta)$

$r = 10 + 4(\cos\pi \cos \theta - \sin \pi \sin \theta)$

$r = 10 - 4 \cos \theta$ Not an equivalent equation

Answer: Symmetric with respect to polar axis.

9. $r = \dfrac{6}{1 + \sin\theta}$

$\theta = \dfrac{\pi}{2}$: $r = \dfrac{6}{1 + \sin(\pi - \theta)}$

$r = \dfrac{6}{1 + \sin\pi\cos\theta - \cos\pi\sin\theta}$

$r = \dfrac{6}{1 + \sin\theta}$

Equivalent equation

Polar axis: $r = \dfrac{6}{1 + \sin(-\theta)}$

$r = \dfrac{6}{1 - \sin\theta}$

Not an equivalent equation

$-r = \dfrac{6}{1 + \sin(\pi - \theta)}$

$-r = \dfrac{6}{1 + \sin\theta}$

Not an equivalent equation

The pole: $-r = \dfrac{6}{1 + \sin\theta}$

Not an equivalent equation

$r = \dfrac{6}{1 + \sin(\pi + \theta)}$

$r = \dfrac{6}{1 - \sin\theta}$

Not an equivalent equation

Answer: Symmetric with respect to $\theta = \dfrac{\pi}{2}$.

11. $r = 6\sin\theta$

$\theta = \dfrac{\pi}{2}$: $-r = 6\sin(-\theta)$

$r = 6\sin\theta$

Equivalent equation

Polar axis: $r = 6\sin(-\theta)$

$r = -6\sin\theta$

Not an equivalent equation

$-r = 6\sin(\pi - \theta)$

$-r = 6(\sin\pi\cos\theta - \cos\pi\sin\theta)$

$-r = 6\sin\theta$

Not an equivalent equation

Pole: $-r = 6\sin\theta$

Not an equivalent equation

$r = 6\sin(\pi + \theta)$

$r = -6\sin\theta$

Not an equivalent equation

Answer: Symmetric with respect to $\theta = \dfrac{\pi}{2}$.

13. $r = 4\sec\theta\csc\theta$

$\theta = \dfrac{\pi}{2}$: $-r = 4\sec(-\theta)\csc(-\theta)$

$-r = -4\sec\theta\csc\theta$

$r = 4\sec\theta\csc\theta$

Equivalent equation

Polar axis: $-r = 4\sec(\pi - \theta)\csc(\pi - \theta)$

$-r = 4(-\sec\theta)\csc\theta$

$r = 4\sec\theta\csc\theta$

Equivalent equation

Pole: $r = 4\sec(\pi + \theta)\csc(\pi + \theta)$

$r = 4(-\sec\theta)(-\csc\theta)$

$r = 4\sec\theta\csc\theta$

Equivalent equation

Answer: Symmetric with respect to $\theta = \pi/2$, pole axis, and pole

15. $r^2 = 25 \sin 2\theta$

$\theta = \frac{\pi}{2}$: $(-r)^2 = 25 \sin(2(-\theta))$

$r^2 = -25 \sin 2\theta$ Not an equivalent equation

$r^2 = 25 \sin(2(\pi - \theta))$

$r^2 = 25 \sin(2\pi - 2\theta)$

$r^2 = 25(\sin 2\pi \cos 2\theta - \cos 2\pi \sin 2\theta)$

$r^2 = -25 \sin 2\theta$ Not an equivalent equation

Polar axis: $r^2 = 25 \sin(2(-\theta))$

$r^2 = -25 \sin 2\theta$ Not an equivalent equation

$(-r)^2 = 25 \sin(2(\pi - \theta))$

$r^2 = -25 \sin 2\theta$ Not an equivalent equation

Pole: $(-r)^2 = 25 \sin(2\theta)$

$r^2 = 25 \sin 2\theta$ Equivalent equation

Answer: Symmetric with respect to pole.

17. $|r| = |10(1 - \sin \theta)|$

$= 10|1 - \sin \theta| \leq 10(2) = 20$

$|1 - \sin \theta| = 2$

$1 - \sin \theta = 2$ or $1 - \sin \theta = -2$

$\sin \theta = -1$ $\quad$ $\sin \theta = 3$

$\theta = \frac{3\pi}{2}$ $\quad$ Not possible

Maximum: $|r| = 20$ when $\theta = \frac{3\pi}{2}$.

$r = 0$ when $1 - \sin \theta = 0$

$\sin \theta = 1$

$\theta = \frac{\pi}{2}$.

19. $|r| = |4 \cos 3\theta| = 4|\cos 3\theta| \leq 4$

$|\cos 3\theta| = 1$

$\cos 3\theta = \pm 1$

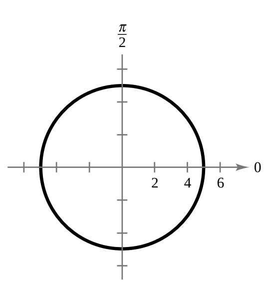
$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

Maximum: $|r| = 4$ when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

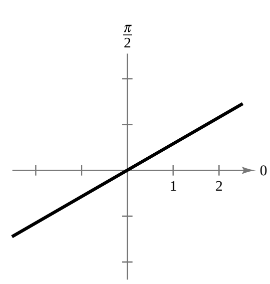
$r = 0$ when $\cos 3\theta = 0$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

21. Circle: $r = 5$



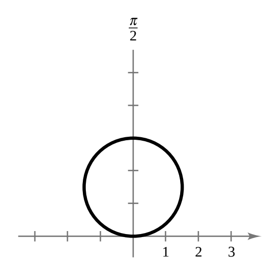
23. $r = \frac{\pi}{6}$ is a circle.



25. $r = 3 \sin \theta$

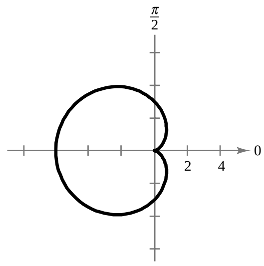
Symmetric with respect to $\theta = \frac{\pi}{2}$

Circle with radius of $\frac{3}{2}$



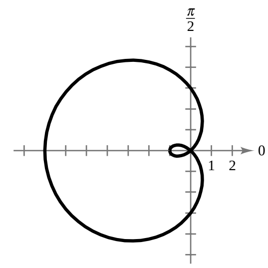
27. $r = 3(1 - \cos \theta)$

Cardioid



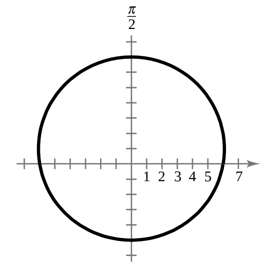
29. $r = 3 - 4 \cos \theta$

Limaçon



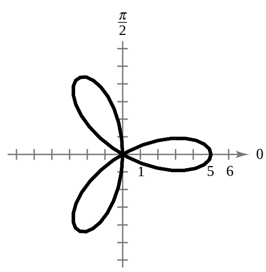
31. $r = 6 + \sin \theta$

Convex limaçon



33. $r = 5 \cos 3\theta$

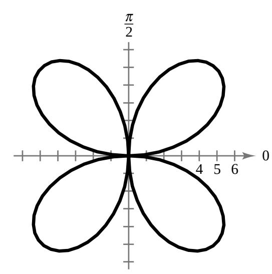
Rose curve



35. $r = 7 \sin 2\theta$

Rose curve

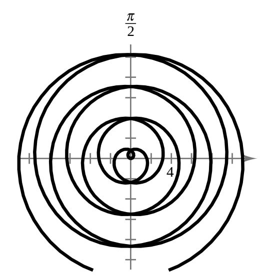
4 petals



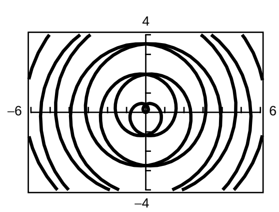
37. $r = \dfrac{\theta}{2}$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

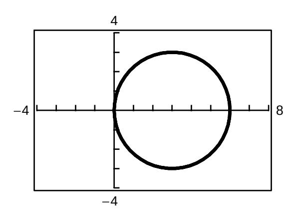
Spiral



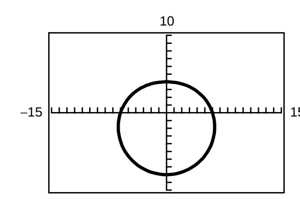
39. $-10\pi \le \theta \le 10\pi$



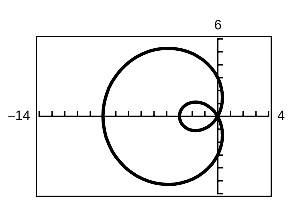
41. $0 \le \theta \le \pi$



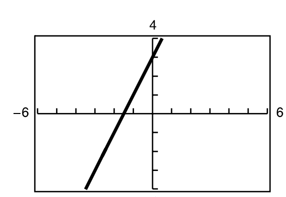
43. $0 \le \theta \le 2\pi$



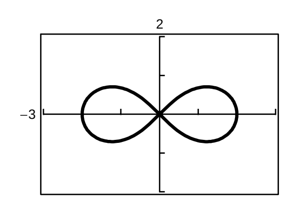
45. $0 \le \theta \le 2\pi$



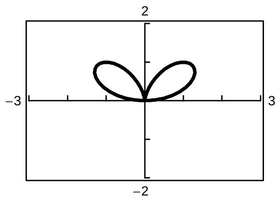
47. $0 \le \theta \le \dfrac{\pi}{2}$



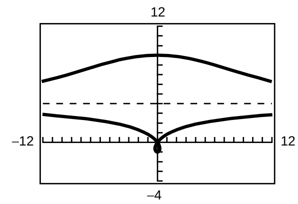
49. $-2\pi \le \theta \le 2\pi$



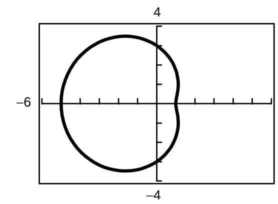
51. $0 \le \theta \le \pi$



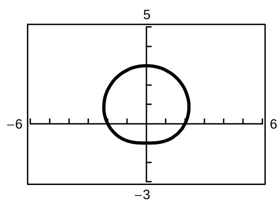
53. $0 \le \theta < 2\pi$



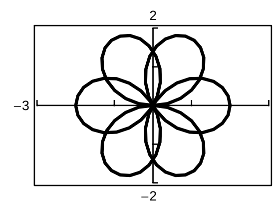
55. $r = 3 - 2\cos\theta,\ 0 \le \theta < 2\pi$



57. $r = 2 + \sin\theta,\ 0 \le \theta < 2\pi$

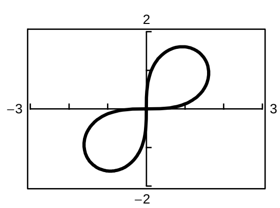


59. $r = 2\cos\left(\dfrac{3\theta}{2}\right),\ 0 \le \theta < 4\pi$



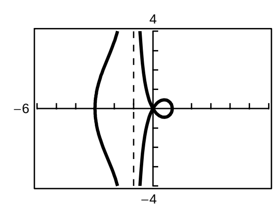
61. $r^2 = 4\sin 2\theta,\ 0 \le \theta < \dfrac{\pi}{2}$

$\left(\text{Use } r_1 = \sqrt{4\sin 2\theta} \text{ and } r_2 = -\sqrt{4\sin 2\theta}.\right)$



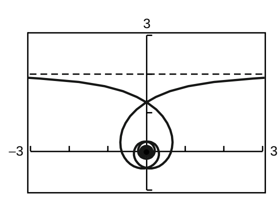
63. $r = 2 - \sec\theta$

$x = -1$ is an asymptote.



65. $r = \dfrac{2}{\theta}$

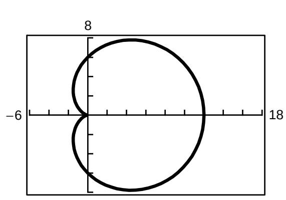
$y = 2$ is an asymptote.



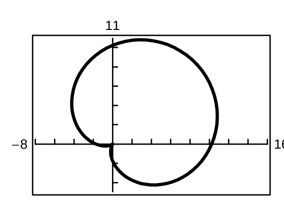
67. False. If $\theta = \dfrac{11\pi}{6}$, $r = 4$

69. False. It has 5 petals

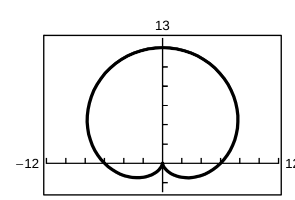
71. (a)



(b)



(c)



The angle ϕ rotates the graph around the pole. In part (c), $r = 6\left[1 + \cos\left(\theta - \dfrac{\pi}{2}\right)\right] = 6[1 + \sin\theta]$.

73. Use the result of Exercise 72.

(a) Rotation: $\phi = \frac{\pi}{2}$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f\left(\sin\left(\theta - \frac{\pi}{2}\right)\right) = f(-\cos\theta)$

(b) Rotation: $\phi = \pi$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f(\sin(\theta - \pi)) = f(-\sin\theta)$

(c) Rotation: $\phi = \pi$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f\left(\sin\left(\theta - \frac{3\pi}{2}\right)\right) = f(\cos\theta)$

75. (a) $r = 2\sin\left[2\left(\theta - \frac{\pi}{6}\right)\right]$

$= 4\sin\left(\theta - \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right)$

$= 2\sin\left(2\theta - \frac{\pi}{3}\right)$

$= \sin 2\theta - \sqrt{3}\cos 2\theta$

(b) $r = 2\sin\left[2\left(\theta - \frac{\pi}{2}\right)\right]$

$= 2\sin(2\theta - \pi)$

$= -2\sin 2\theta$

$= -4\sin\theta\cos\theta$

(c) $r = 2\sin\left[2\left(\theta - \frac{2\pi}{3}\right)\right]$

$= 4\sin\left(\theta - \frac{2\pi}{3}\right)\cos\left(\theta - \frac{2\pi}{3}\right)$

$= 2\sin\left(2\theta - \frac{4\pi}{3}\right)$

$= \sqrt{3}\cos 2\theta - \sin 2\theta$

(d) $r = 2\sin[2(\theta - \pi)]$

$= 2\sin(2\theta - 2\pi)$

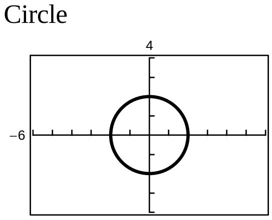
$= 2\sin 2\theta$

$= 4\sin\theta\cos\theta$

77. $r = 2 + k\cos\theta$

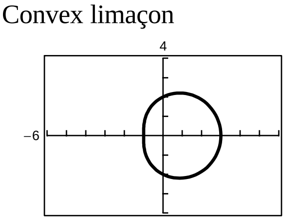
$k = 0$

Circle



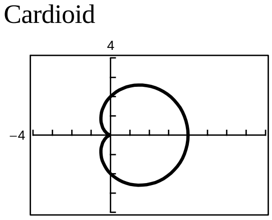
$k = 1$

Convex limaçon



$k = 2$

Cardioid



$k = 3$

Limaçon with inner loop