Section 6.2 Law of Cosines

- If ABC is any oblique triangle with sides a, b, and c, then the Law of Cosines says:
  (a) $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
  (b) $b^2 = a^2 + c^2 - 2ac\cos B$ or $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
  (c) $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
- You should be able to use the Law of Cosines to solve an oblique triangle for the remaining three parts, given:
  (a) Three sides (SSS)
  (b) Two sides and their included angle (SAS)
- Given any triangle with sides of length a, b, and c, then the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}. \qquad \text{(Heron's Formula)}$$

Solutions to Odd-Numbered Exercises

1. Given: $a = 6,\ b = 8,\ c = 12$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 144 - 36}{2(8)(12)} \approx 0.8958 \implies A \approx 26.4^\circ$$

$$\sin B = \frac{b\sin A}{a} \approx \frac{8\sin 26.4^\circ}{6} \approx 0.5928 \implies B \approx 36.3^\circ$$

$$C \approx 180^\circ - 26.4^\circ - 36.3^\circ = 117.3^\circ$$

3. Given: $A = 50^\circ, b = 15, c = 30$

$$a^2 = b^2 + c^2 - 2bc\cos A = 225 + 900 - 2(15)(30)\cos 50^\circ \approx 546.49 \implies a \approx 23.4$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \approx 0.8708 \implies B \approx 29.5^\circ$$

$$C = 180^\circ - A - B = 180^\circ - 50^\circ - 29.5^\circ = 100.5^\circ$$

5. Given: $a = 9,\ b = 12,\ c = 15$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{81 + 144 - 225}{2(9)(12)} = 0 \implies C = 90^\circ$$

$$\sin A = \frac{9}{15} = \frac{3}{5} \implies A \approx 36.9^\circ$$

$$B \approx 180^\circ - 90^\circ - 36.9^\circ = 53.1^\circ$$

7. Given: $a = 75.4, b = 48, c = 48$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{48^2 + 48^2 - 75.4^2}{2(48)(48)} = -0.2338 \implies A \approx 103.5^\circ$$

$$\sin B = \frac{b \sin A}{a} = \frac{48 \sin(103.5^\circ)}{75.4} = 0.6190 \implies B \approx 38.2$$

$C = B \approx 38.2$ (Because of roundoff error, $A + B + C \neq 360^\circ$)

9. Given: $B = 8^\circ\, 15' = 8.25^\circ, a = 26, c = 18$

$$b^2 = a^2 + c^2 - 2ac \cos B = 26^2 + 18^2 - 2(26)(18)\cos(8.25) \approx 73.6863 \implies b \approx 8.6$$

$$\sin C = \frac{c \sin B}{b} = \frac{18 \sin(8.25)}{8.6} \approx 0.3 \implies C \approx 17.5^\circ$$

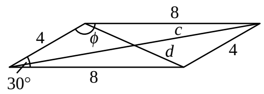
$$A = 180^\circ - B - C = 180^\circ - 8.25^\circ - 17.5^\circ = 154.25$$

11. $d^2 = 4^2 + 8^2 - 2(4)(8)\cos 30^\circ \approx 24.57 \implies d \approx 4.96$

$2\phi = 360^\circ - 2\theta \implies \phi = 150^\circ$

$c^2 = 4^2 + 8^2 - 2(4)(8)\cos 150^\circ \approx 135.43$

$c \approx 11.64$



13. $\cos \phi = \dfrac{10^2 + 14^2 - 20^2}{2(10)(14)}$

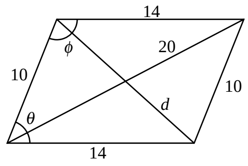
$\phi \approx 111.8$

$2\theta \approx 360^\circ - 2(111.80^\circ)$

$\theta = 68.2^\circ$

$d^2 = 10^2 + 14^2 - 2(10)(14)\cos 68.2^\circ$

$d \approx 13.86$



15. $\cos \alpha = \dfrac{(9)^2 + (10)^2 - (6)^2}{2(9)(10)}$

$\alpha = 36.3^\circ$

$\cos \beta = \dfrac{6^2 + 10^2 - 9^2}{2(6)(10)}$

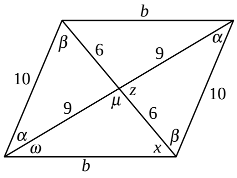
$\beta \approx 62.7^\circ$

$z = 180^\circ - \alpha - \beta \approx 80.9$

$\mu = 180^\circ - z \approx 99.1$

$b^2 = 9^2 + 6^2 - 2(9)(6)(\cos 99.0^\circ)$

$b \approx 11.58$



$\cos \omega = \dfrac{9^2 + 11.58^2 - 6^2}{2(9)(11.58)}$

$\omega \approx 30.8^\circ$

$\theta = \alpha + \omega \approx 67.1^\circ$

$\cos x = \dfrac{6^2 + 11.58^2 - 9^2}{2(6)(11.58)}$

$x \approx 50.1$

$\phi = \beta + x \approx 112.8^\circ$

17. Given: $a = 5, b = 9, c = 10$

$$s = \frac{a + b + c}{2} = 12$$

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{12(7)(3)(2)} \approx 22.45 \text{ square units}$$

19. Given: $a = 3.5, b = 10.2, c = 9$

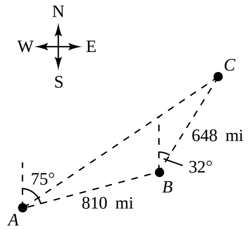
$$s = \frac{a + b + c}{2} = 11.35$$

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{11.35(7.85)(1.15)(2.35)} \approx 15.52 \text{ square units}$$

21. $a = 20,\ b = 20,\ c = 10 \implies s = \dfrac{20 + 20 + 10}{2} = 25$

$$\text{Area} = \sqrt{25(5)(5)(15)} \approx 96.82$$

23.



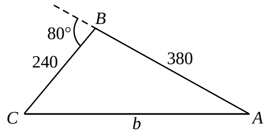
$$B = 105° + 32° = 137°$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$= 648^2 + 810^2 - 2(648)(810)\cos(137°)$$
$$= 1{,}843{,}749.862$$
$$b = 1357.8 \text{ miles}$$

From the Law of Sines, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \implies \sin A = \dfrac{a}{b}\sin B = \dfrac{648}{1357.8}\sin(137°) \approx 0.32548$

$\implies A \approx 19° \implies$ Bearing S 56° W

25. Angle at $B = 180° - 80° = 100°$

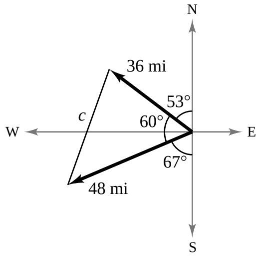
$b^2 = 240^2 + 380^2 - 2(240)(380)\cos 100° \approx 233{,}673.4 \implies b \approx 483.4$ meters



27. $C = 180° - 53° - 67° = 60°$

$c^2 = a^2 + b^2 - 2ab\cos C = 36^2 + 48^2 - 2(36)(48)(0.5) = 1872$

$c \approx 43.3$ mi



29. (a) $\cos\theta = \dfrac{273^2 + 178^2 - 235^2}{2(273)(178)}$

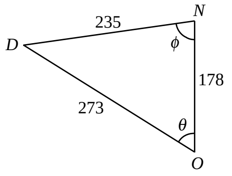
$\theta \approx 58.4°$

Bearing: N 58.4° W

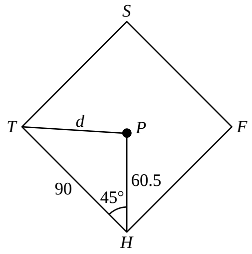
(b) $\cos\phi = \dfrac{235^2 + 178^2 - 273^2}{2(235)(178)}$

$\phi \approx 81.5°$

Bearing: S 81.5° W



31. $d^2 = 60.5^2 + 90^2 - 2(60.5)(90)\cos 45° \approx 4059.9 \implies d \approx 63.7$ ft



33. $\overline{RS} = \sqrt{8^2 + 10^2} = \sqrt{164} = 2\sqrt{41} \approx 12.8$ ft

$\overline{PQ} = \frac{1}{2}\sqrt{16^2 + 10^2} = \frac{1}{2}\sqrt{356} = \sqrt{89} \approx 9.4$ ft

$\tan P = \frac{10}{16}$

$P = \arctan\frac{5}{8} \approx 32.0°$

$\overline{QS} = \sqrt{8^2 + 9.4^2 - 2(8)(9.4)\cos 32°} \approx \sqrt{24.81} \approx 5.0$ ft

35. (a) $7^2 = 1.5^2 + x^2 - 2(1.5)(x)\cos\theta$

$49 = 2.25 + x^2 - 3x\cos\theta$

(b) $x^2 - 3x\cos\theta = 46.75$

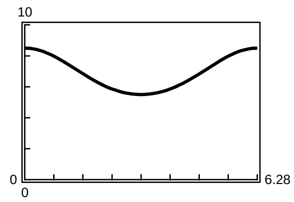
$$x^2 - 3x\cos\theta + \left(\frac{3\cos\theta}{2}\right)^2 = 46.75 + \left(\frac{3\cos\theta}{2}\right)^2$$

$$\left[x - \frac{3\cos\theta}{2}\right]^2 = \frac{187}{4} - \frac{9\cos^2\theta}{4}$$

$$x - \frac{3\cos\theta}{2} = \pm\sqrt{\frac{187 + 9\cos^2\theta}{4}}$$

Choosing the positive values of x, we have
$x = \frac{1}{2}\left(3\cos\theta + \sqrt{9\cos^2\theta + 187}\right)$.

(c)



(d) $x = \dfrac{1}{2}\left(3\cos\pi + \sqrt{9\cos^2\pi + 187}\right)$

$= 5.5$

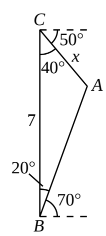
≈ 6 inches

37. $A = 180° - 40° - 20° = 120°$

$\dfrac{x}{\sin 20°} = \dfrac{7}{\sin 120°}$

$x = \dfrac{7\sin 20°}{\sin 120°}$

$x = 2.76$ feet



39. False. This is not a triangle! $5 + 10 < 16$

41. (a) Working with $\triangle OBC$, we have $\cos\alpha = \dfrac{a/2}{R}$.

This implies that $2R = a/\cos\alpha$. Since we know that

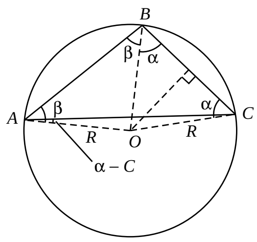
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we can complete the proof by showing that $\cos\alpha = \sin A$. The solution of the system

$$A + B + C = 180°$$
$$\alpha - C + A = \beta$$
$$\alpha + \beta = B$$

is $\alpha = 90° - A$. Therefore:

$$2R = \frac{a}{\cos\alpha} = \frac{a}{\cos(90° - A)} = \frac{a}{\sin A}.$$



(b) By Heron's Formula, the area of the triangle is

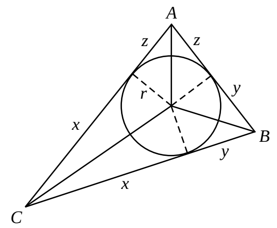
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

We can also find the area by dividing the area into six triangles and using the fact that the area is 1/2 the base times the height. Using the figure as given, we have

$$\text{Area} = \frac{1}{2}xr + \frac{1}{2}xr + \frac{1}{2}yr + \frac{1}{2}yr + \frac{1}{2}zr + \frac{1}{2}zr$$
$$= r(x + y + z)$$
$$= rs.$$

Therefore: $rs = \sqrt{s(s-a)(s-b)(s-c)} \implies$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$



43. Given: $a = 200$ ft, $b = 250$ ft, $c = 325$ ft

$$s = \frac{200 + 250 + 325}{2} \approx 387.5$$

Radius of the inscribed circle: $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\dfrac{(187.5)(137.5)(62.5)}{387.5}} \approx 64.5$ ft

Circumference of an inscribed circle: $C = 2\pi r \approx 2\pi(64.5) \approx 405$ ft

45. $3\sec x + 4 = 10$

$$\sec x = 2$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

47. $\dfrac{3}{\cos x \sin x} = 2\sqrt{3}\csc x = 2\sqrt{3}\left(\dfrac{1}{\sin x}\right)$

$$\frac{3}{\cos x} = 2\sqrt{3}$$
$$\cos x = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{6}, \frac{11\pi}{6}$$

49. $\cos\dfrac{5\pi}{6} - \cos\dfrac{\pi}{3} = -2\sin\left(\dfrac{\frac{5\pi}{6} + \frac{\pi}{3}}{2}\right)\sin\left(\dfrac{\frac{5\pi}{6} - \frac{\pi}{3}}{2}\right)$

$$= -2\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{4}\right)$$