Section 6.4 Vectors and Dot Products

- Know the definition of the dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$.

  $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$
- Know the following properties of the dot product:
  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2. $\mathbf{0} \cdot \mathbf{v} = 0$
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
  5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
- If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

  $$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$
- The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- Know the definition of vector components. $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal, and $\mathbf{w}_1$ is parallel to $\mathbf{v}$. $\mathbf{w}_1$ is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ and is denoted by

  $$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

  Then we have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.
- Know the definition of work.
  1. Projection form: $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$
  2. Dot product form: $W = \mathbf{F} \cdot \overrightarrow{PQ}$

Solutions to Odd-Numbered Exercises

1. $\mathbf{u} = \langle 3, 6 \rangle, \mathbf{v} = \langle 2, -4 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + 6(-4) = -18$

3. $\mathbf{u} = 4\mathbf{i} - 7\mathbf{j}, \mathbf{v} = \mathbf{i} - \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 4(1) + (-7)(-1) = 11$

5. $\mathbf{u} = \langle 2, 2 \rangle$

$\mathbf{u} \cdot \mathbf{u} = 2(2) + 2(2) = 8$

The result is a scalar.

7. $\mathbf{u} = \langle 2, 2 \rangle, \mathbf{v} = \langle -3, 4 \rangle, w = \langle 1, -4 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 2(-3) + 2(4) = 2$

$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2\langle 1, -4 \rangle = \langle 2, -8 \rangle$, vector

9. $\mathbf{u} = \langle 2, 2 \rangle, \mathbf{v} = \langle -3, 4 \rangle$

$\mathbf{u} \cdot 2\mathbf{v} = 2\mathbf{u} \cdot \mathbf{v} = 2(2) = 4$, scalar

11. $\mathbf{u} = \langle -5, 12 \rangle$

$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(-5)^2 + 12^2} = 13$

13. $\mathbf{u} = 20\mathbf{i} + 25\mathbf{j}$

$\|\mathbf{u}\| = \sqrt{(20)^2 + (25)^2} = \sqrt{1025} = 5\sqrt{41}$

15. $\mathbf{u} = 6\mathbf{j}$

$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{6(6)} = 6$

17. $\mathbf{u} = \langle -1, 0 \rangle, \mathbf{v} = \langle 0, 2 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{(1)(2)} = 0 \implies \theta = 90^\circ$$

19. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-6+12}{(5)(\sqrt{13})} = \frac{6}{5\sqrt{13}}$$

$$\theta = \arccos\left(\frac{6}{5\sqrt{13}}\right) \approx 70.56°$$

21. $\mathbf{u} = 2\mathbf{i}, \mathbf{v} = -3\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{(2)(3)} = 0 \implies \theta = 90°$$

23. $$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\mathbf{v} = \left(\cos\frac{3\pi}{4}\right)\mathbf{i} + \left(\sin\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

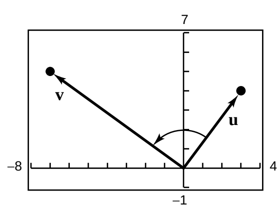
$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \mathbf{u}\cdot\mathbf{v} = \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{2}+\sqrt{6}}{4}$$

$$\theta = \arccos\left(\frac{-\sqrt{2}+\sqrt{6}}{4}\right) = 75° = \frac{5\pi}{12}$$

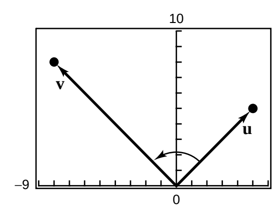
25. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j},\ \mathbf{v} = -7\mathbf{i} + 5\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = -\frac{1}{(5)(\sqrt{74})} \implies \theta \approx 91.33°$$



27. $\mathbf{u} = 5\mathbf{i} + 5\mathbf{j},\ \mathbf{v} = -8\mathbf{i} + 8\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = 0 \implies \theta = 90°$$



29. $P = (1, 2),\ Q = (3, 4),\ R = (2, 5)$

$$\overrightarrow{PQ} = \langle 2, 2\rangle,\ \overrightarrow{PR} = \langle 1, 3\rangle,\ \overrightarrow{QR} = \langle -1, -1\rangle$$

$$\cos\alpha = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\,\|\overrightarrow{PR}\|} = \frac{8}{(2\sqrt{2})(\sqrt{10})} \implies \alpha = \arccos\frac{2}{\sqrt{5}} \approx 26.6°$$

$$\cos\beta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{QR}}{\|\overrightarrow{PQ}\|\,\|\overrightarrow{QR}\|} = 0 \implies \beta = 90°.$$ Thus, $\gamma = 180° - 26.6° - 90° = 63.4°$.

31. $$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

$$= (4)(10)\cos\frac{2\pi}{3}$$

$$= 40\left(-\frac{1}{2}\right)$$

$$= -20$$

33. $\mathbf{u} = \langle -12, 30\rangle,\ \mathbf{v} = \left\langle \frac{1}{2}, -\frac{5}{4}\right\rangle$

$\mathbf{u} = -24\mathbf{v} \implies$ **u** and **v** are parallel.

35. $\mathbf{u} = \frac{1}{4}(3\mathbf{i} - \mathbf{j}),\ \mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

$\mathbf{u} \neq k\mathbf{v} \implies$ Not parallel

$\mathbf{u} \cdot \mathbf{v} \neq 0 \implies$ Not orthogonal

Neither

37. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j},\ \mathbf{v} = -\mathbf{i} - \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ **u** and **v** are orthogonal.

39. $\mathbf{u} = \langle 3, 4\rangle,\ \mathbf{v} = \langle 8, 2\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{32}{68}\right)\mathbf{v} = \frac{8}{17}\langle 8,\ 2\rangle = \frac{16}{17}\langle 4, 1\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 3, 4\rangle - \frac{16}{17}\langle 4, 1\rangle = \frac{13}{17}\langle -1, 4\rangle$$

41. $\mathbf{u} = \langle 0, 3\rangle,\ \mathbf{v} = \langle 2, 15\rangle$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \frac{45}{229}\langle 2, 15\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3\rangle - \frac{45}{229}\langle 2, 15\rangle = \left\langle -\frac{90}{229}, \frac{12}{229}\right\rangle = \frac{6}{229}\langle -15, 2\rangle$$

43. $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$ since they are parallel.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{18 + 18}{36 + 16}\mathbf{v} = \frac{26}{52}\langle 6, 4\rangle = \langle 3, 2\rangle = \mathbf{u}$$

45. $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{0}$ since they are perpendicular.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \mathbf{0}, \text{ since } \mathbf{u} \cdot \mathbf{v} = 0.$$

47. $\mathbf{u} = \langle 4, 7\rangle$

For **v** to be orthogonal to **u**, $\mathbf{u} \cdot \mathbf{v}$ must be equal 0.
Two possibilities: $\langle 7, -4\rangle$ and $\langle -7, 4\rangle$

49. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j}$

For **v** to be orthogonal to **u**, $\mathbf{u} \cdot \mathbf{v}$ must equal 0.
Two possibilities: $\left\langle \frac{3}{4}, \frac{1}{2}\right\rangle$ and $\left\langle -\frac{3}{4}, -\frac{1}{2}\right\rangle$

51. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{v}\|\,\|\overrightarrow{PQ}\|$ where $\overrightarrow{PQ} = \langle 4, 7\rangle$ and $\mathbf{v} = \langle 1, 4\rangle$.

$$\text{proj}_{\overrightarrow{PQ}}\mathbf{v} = \left(\frac{\mathbf{v} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PQ}\|^2}\right)\overrightarrow{PQ} = \left(\frac{32}{65}\right)\langle 4, 7\rangle$$

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{v}\|\,\|\overrightarrow{PQ}\| = \left(\frac{32\sqrt{65}}{65}\right)\left(\sqrt{65}\right) = 32$$

53. $\mathbf{u} = \langle 1245, 2600\rangle,\ \mathbf{v} = \langle 12.20, 8.50\rangle$

$\mathbf{u} \cdot \mathbf{v} = 1245(12.20) + 2600(8.50) = \$37{,}289$

This gives the total revenue that can be earned by selling all of the units.

55. (a) $\mathbf{F} = -36{,}000\mathbf{j}$ Gravitational force

$\mathbf{v} = (\cos 10°)\mathbf{i} + (\sin 10°)\mathbf{j}$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v} \approx -6251.3\mathbf{v}$$

The magnitude of this force is 6251.3, therefore a force of 6251.3 pounds is needed to keep the truck from rolling down the hill.

(b) $\mathbf{w}_2 = \mathbf{F} - \mathbf{w}_1 = -36{,}000\mathbf{j} + 6251.3\ (\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$

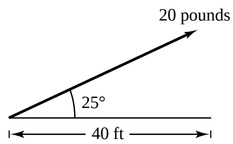
$= [(6251.3 \cos 10°)\mathbf{i} + (6251.3 \sin 10° - 36{,}000)\mathbf{j}]$

$\|\mathbf{w}_2\| \approx 35{,}453.1$ pounds

57. $W = (245)(3) = 735$ Newton-meters

59. $W = (\cos 30°)(45)(20) \approx 779.4$ foot-pounds

61. $W = (\cos 25°)(20)(40) \approx 725.05$ ft/pounds



63. True. $\mathbf{u} \cdot \mathbf{v} = 0$

65. $\mathbf{u} \cdot \mathbf{v} = 0 \implies$ they are orthogonal(unit vectors)

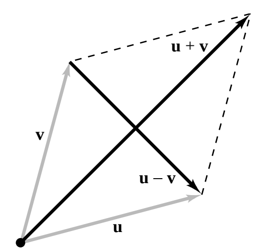
67. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{u} \implies \mathbf{u}$ and $\mathbf{v}$ are parallel

(b) $\text{proj}_{\mathbf{v}}\,\mathbf{u} = 0 \implies \mathbf{u}$ and $\mathbf{v}$ are orthogonal

69. Let $\mathbf{u}$ and $\mathbf{v}$ be two sides of the rhombus $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \\ &= 0\end{aligned}$$

Hence, the diagonals are perpendicular.



71. (a) $\mathbf{0} \cdot \mathbf{v} = \langle 0, 0\rangle \cdot \langle v_1, v_2\rangle = 0v_1 + 0v_2 = 0$

(b)
$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \\ &= \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle + \langle u_1, u_2\rangle \cdot \langle w_1, w_2\rangle \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}\end{aligned}$$

73.
$$\begin{aligned}2\cos(x + \pi) + 2\cos(x - \pi) &= 0 \\ -2\cos x - 2\cos x &= 0 \\ -4\cos x &= 0 \\ \cos x &= 0 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2}\end{aligned}$$

75.
$$\sin\left(x - \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{3}\right) = \frac{3}{2}$$

$$\begin{aligned}\left[\sin x \cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3}\right] - \left[\sin x \cdot \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}\right] &= \frac{3}{2} \\ (-2\cos x)\left(\frac{\sqrt{3}}{2}\right) &= \frac{3}{2} \\ \cos x &= -\frac{\sqrt{3}}{2} \\ x &= \frac{5\pi}{6}, \frac{7\pi}{6}\end{aligned}$$

77. $s = \dfrac{a+b+c}{2} = \dfrac{8+15+16}{2} = 19.5$

Area $= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{19.5(19.5-8)(19.5-15)(19.5-16)} \approx 59.43$ sq. units

79. (a) $\mathbf{u} + \mathbf{v} = \langle 3, 0\rangle + \langle 4, -1\rangle = \langle 7, -1\rangle$

(b) $2\mathbf{v} - \mathbf{u} = 2\langle 4, -1\rangle - \langle 3, 0\rangle = \langle 5, -2\rangle$

(c) $3\mathbf{u} - 5\mathbf{v} = 3\langle 3, 0\rangle - 5\langle 4, -1\rangle = \langle -11, 5\rangle$

81. (a) $\mathbf{u} + \mathbf{v} = \langle -2, -2\rangle + \langle -4, 5\rangle = \langle -6, 3\rangle$

(b) $2\mathbf{v} - \mathbf{u} = 2\langle -4, 5\rangle - \langle -2, -2\rangle = \langle -6, 12\rangle$

(c) $3\mathbf{u} - 5\mathbf{v} = 3\langle -2, -2\rangle - 5\langle -4, 5\rangle = \langle 14, -31\rangle$

83. The car will cost $(1.04)(23{,}500) = \$24{,}440$ in one month, \$940 over the present price. This is more than the \$725 interest penalty. Buy now.

85. Let x be the number of people presently in the group. Each share is $\dfrac{250{,}000}{x}$.

Also, $\dfrac{250{,}000}{x} - 6250 = \dfrac{250{,}000}{x+2}$

Solving this equation, $x = 8$.

Section 6.5 Trigonometric Form of a Complex Number

- You should be able to graphically represent complex numbers.
- The absolute value of the complex numbers $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
- The trigonometric form of the complex number $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$ where
  (a) $a = r\cos\theta$
  (b) $b = r\sin\theta$
  (c) $r = \sqrt{a^2 + b^2}$; r is called the modulus of z.
  (d) $\tan\theta = b/a$; θ is called the argument of z.
- Given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:
  (a) $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
  (b) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0$
- You should know DeMoivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$, then for any positive integer n,
  $z^n = r^n(\cos n\theta + i\sin n\theta)$.
- You should know that for any positive integer n, $z = r(\cos\theta + i\sin\theta)$ has n distinct nth roots given by
  $$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$
  where $k = 0, 1, 2, \ldots, n-1$.