Section 9.3 Geometric Sequences and Series

- You should be able to identify a geometric sequence, find its common ratio, and find the nth term.
- You should be able to find the nth partial sum of a geometric sequence with common ratio r using the formula.

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$$

- You should know that if $|r| < 1$, then

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1 - r}.$$

Solutions to Odd-Numbered Exercises

1. $5, 15, 45, 135, \ldots$

Geometric sequence, $r = 3$

3. $6, 18, 30, 42, \ldots$

Not a geometric sequence

(Note: It is an arithmetic sequence with $d = 12$.)

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

Geometric sequence, $r = -\frac{1}{2}$

7. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Not a geometric sequence

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Not a geometric sequence

11. $a_1 = 8, r = 3$

$a_2 = 8(3) = 24$

$a_3 = 24(3) = 72$

$a_4 = 72(3) = 216$

$a_5 = 216(3) = 648$

13. $a_1 = 1, \ r = \frac{1}{2}$

$a_1 = 1$

$a_2 = 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$a_3 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$a_4 = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$

$a_5 = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16}$

15. $a_1 = 5, \ r = -\frac{1}{10}$

$a_1 = 5$

$a_2 = 5\left(-\frac{1}{10}\right) = -\frac{1}{2}$

$a_3 = \left(-\frac{1}{2}\right)\left(-\frac{1}{10}\right) = \frac{1}{20}$

$a_4 = \frac{1}{20}\left(-\frac{1}{10}\right) = -\frac{1}{200}$

$a_5 = \left(-\frac{1}{200}\right)\left(-\frac{1}{10}\right) = \frac{1}{2000}$

17. $a_1 = 3.5, r = 5$

$a_2 = 3.5(5) = 17.5$

$a_3 = 17.5(5) = 87.5$

$a_4 = 87.5(5) = 437.5$

$a_5 = 437.5(5) = 2187.5$

19. $a_1 = 1, \ r = e$

$a_1 = 1$

$a_2 = 1(e) = e$

$a_3 = (e)(e) = e^2$

$a_4 = (e^2)(e) = e^3$

$a_5 = (e^3)(e) = e^4$

21. $a_1 = 64, \ a_{k+1} = \frac{1}{2} a_k$

$a_1 = 64$

$a_2 = \frac{1}{2}(64) = 32$

$a_3 = \frac{1}{2}(32) = 16$

$a_4 = \frac{1}{2}(16) = 8$

$a_5 = \frac{1}{2}(8) = 4$

$r = \frac{1}{2}, a_n = 64\left(\frac{1}{2}\right)^{n-1} = 128\left(\frac{1}{2}\right)$

23. $a_1 = 4,\ a_{k+1} = 3a_k$

$a_1 = 4$

$a_2 = 3(4) = 12$

$a_3 = 3(12) = 36$

$a_4 = 3(36) = 108$

$a_5 = 3(108) = 324$

$r = 3,\ a_n = 4(3)^{n-1} = \frac{4}{3}(3)^n$

25. $a_k = 6,\ a_{k+1} = -\frac{3}{2}a_k$

$a_1 = 6$

$a_2 = -\frac{3}{2}(6) = -9$

$a_3 = -\frac{3}{2}(-9) = \frac{27}{2}$

$a_4 = -\frac{3}{2}\left(\frac{27}{2}\right) = -\frac{81}{4}$

$a_5 = -\frac{3}{2}\left(-\frac{81}{4}\right) = \frac{243}{8}$

$r = -\frac{3}{2},\ a_n = 6\left(-\frac{3}{2}\right)^{n-1}$

27. $a_1 = 4,\ r = \frac{1}{2},\ n = 10$

$a_n = a_1 r^{n-1}$

$a_{10} = 4\left(\frac{1}{2}\right)^9 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$

29. $a_1 = 6,\ r = -\dfrac{1}{3},\ n = 12$

$a_n = a_1 r^{n-1}$

$a_{12} = 6\left(-\dfrac{1}{3}\right)^{11} = \dfrac{-2}{3^{10}}$

31. $a_1 = 500,\ r = 1.02,\ n = 14$

$a_n = a_1 r^{n-1}$

$a_{14} = 500(1.02)^{13} \approx 646.8$

33. $a_1 = 16,\ a_4 = \dfrac{27}{4},\ n = 3$

$\dfrac{27}{4} = 16r^3 \implies r = \dfrac{3}{4}$

$a_n = a_1 r^{n-1}$

$a_3 = 16\left(\dfrac{3}{4}\right)2 = 9$

35. $a_2 = a_1 r = -18 \implies a_1 = \dfrac{-18}{r}$

$a_5 = a_1 r^4 = (a_1 r)r^3 = -18r^3 = \dfrac{2}{3} \implies r = -\dfrac{1}{3}$

$a_1 = \dfrac{-18}{r} = \dfrac{-18}{-1/3} = 54$

$a_6 = a_1 r^5 = 54\left(\dfrac{-1}{3}\right)^5 = \dfrac{54}{243} = -\dfrac{2}{9}$

37. $r = \frac{21}{7} = 3.$

$a_9 = a_1 r^{9-1} = 7(3)^8 = 45{,}927$

39. $r = \frac{30}{5} = 6$

$a_{10} = a_1 r^{10-1} = 5(6)^9 = 50{,}388{,}480$

41. $r = \dfrac{\frac{3}{4}}{\frac{3}{16}} = 4$

$a_{12} = a_1 r^{12-1} = \dfrac{3}{16}(4)^{11} = 786{,}432$

43. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

$r = \frac{2}{3} < 1$, so the sequence is decreasing.

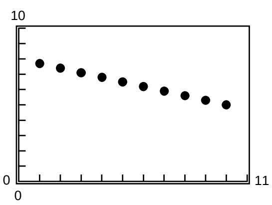
Matches (a).

45. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

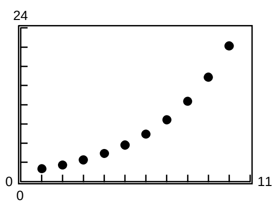
$r = \frac{3}{2} > 1$, so the sequence is increasing.

Matches (b).

47. $a_n = 12(-0.75)^{n-1}$



49. $a_n = 2(1.3)^{n-1}$



51. $8, \ -4, \ 2, \ -1, \ \frac{1}{2}$

$S_1 = 8$

$S_2 = 8 + (-4) = 4$

$S_3 = 8 + (-4) + 2 = 6$

$S_4 = 8 + (-4) + 2 + (-1) = 5$

53. $\sum_{n=1}^{\infty} 16\left(-\frac{1}{2}\right)^{n-1}$

n	1	2	3	4	5	6	7	8	9	10
S_n	16	24	28	30	31	31.5	31.75	31.875	31.9375	31.96875

55. $\sum_{n=1}^{9} 2^{n-1} \implies a_1 = 1, \ r = 2$

$$S_9 = \frac{1(1 - 2^9)}{1 - 2} = 511$$

57. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1} \implies a_1 = 64, \ r = -\frac{1}{2}$

$$S_7 = 64\left[\frac{1 - (-1/2)^7}{1 - (-1/2)}\right] = \frac{128}{3}\left[1 - \left(-\frac{1}{2}\right)^7\right] = 43$$

59. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n = \sum_{n=1}^{21} 3\left(\frac{3}{2}\right)^{n-1} \implies a_1 = 3, \ r = \frac{3}{2}$

$$S_{21} = 3\left[\frac{1 - (3/2)^{21}}{1 - (3/2)}\right] = -6\left[1 - \left(\frac{3}{2}\right)^{21}\right] \approx 29{,}921.31$$

61. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1} \implies a_1 = 8, \ r = -\frac{1}{4}$

$$S_{10} = 8\left[\frac{1 - (-1/4)^{10}}{1 - (-1/4)}\right] = \frac{32}{5}\left[1 - \left(-\frac{1}{4}\right)^{10}\right] \approx 6.4$$

63. $\sum_{n=0}^{5} 300(1.06)^n = \sum_{n=1}^{6} 300(1.06)^{n-1} \implies a_1 = 300, \ r = 1.06$

$$S_6 = 300\left[\frac{1 - (1.06)^6}{1 - 1.06}\right] \approx 2092.60$$

65. $5 + 15 + 45 + \cdots + 3645$

$r = 3$ and $3645 = 5(3)^{n-1} \implies n = 7$

Thus, the sum can be written as $\sum_{n=1}^{7} 5(3)^{n-1}$.

67. $2 - \frac{1}{2} + \frac{1}{8} - \ldots + \frac{1}{2048}$

$r = -\frac{1}{4}$ and $\frac{1}{2048} = 2\left(-\frac{1}{4}\right)^{n-1} \implies n = 7$

$\sum_{n=1}^{7} 2\left(-\frac{1}{4}\right)^{n-1}$

69. $a_1 = 1, \ r = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{a_1}{1 - r} = \frac{1}{1 - (1/2)} = 2$$

71. $a_1 = 1, \ r = -\frac{1}{2}$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1} = \frac{a_1}{1-r} = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

73. $a_1 = 4, \ r = \frac{1}{4}$

$$\sum_{n=0}^{\infty}4\left(\frac{1}{4}\right)^n = \frac{a_1}{1-r} = \frac{4}{1-(1/4)} = \frac{16}{3}$$

75. $\sum_{n=1}^{\infty}2\left(\frac{7}{3}\right)^{n-1}$ does not have a finite sum $\left(\frac{7}{3} > 1\right)$

77. $\sum_{n=0}^{\infty}(0.4)^n$. $a_1 = 1, r = 0.4$

$$\sum_{n=0}^{\infty}(0.4)^n = \frac{a_1}{1-r} = \frac{1}{1-0.4} = \frac{1}{0.6} = \frac{10}{6} = \frac{5}{3}$$

79. $a = -3, r = 0.9$

$$\sum_{n=0}^{\infty} -3(0.9)^n = \frac{a_1}{1-r} = \frac{-3}{1-0.9} = \frac{-3}{0.1} = -30$$

81. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots = \sum_{n=0}^{\infty}8\left(\frac{3}{4}\right)^n = \frac{8}{1-3/4} = 3$

83. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots = \sum_{n=0}^{\infty}3\left(-\frac{1}{3}\right)^n = \frac{a_1}{1-r} = \frac{3}{1-\left(-\frac{1}{3}\right)} = 3\left(\frac{3}{4}\right) = \frac{9}{4}$

85. $0.\overline{36} = \sum_{n=0}^{\infty}0.36(0.01)^n = \frac{0.36}{1-0.01} = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$

87. $0.3\overline{18} = 0.3 + \sum_{n=0}^{\infty}0.018(0.01)^n = \frac{3}{10} + \frac{0.018}{1-0.01}$

$$= \frac{3}{10} + \frac{0.018}{0.99} = \frac{3}{10} + \frac{18}{990} = \frac{3}{10} + \frac{2}{110}$$

$$= \frac{35}{110} = \frac{7}{22}$$

89. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.08}{n}\right)^{n(10)}$

(a) $n = 1,$ $A = 1000(1 + 0.08)^{10} \approx \$2158.92$

(b) $n = 2,$ $A = 1000\left(1 + \frac{0.08}{2}\right)^{2(10)} \approx \$2191.12$

(c) $n = 4,$ $A = 1000\left(1 + \frac{0.08}{4}\right)^{4(10)} \approx \$2208.04$

(d) $n = 12,$ $A = 1000\left(1 + \frac{0.08}{12}\right)^{12(10)} \approx \$2219.64$

(e) $n = 365,$ $A = 1000\left(1 + \frac{0.08}{365}\right)^{365(10)} \approx \$2225.35$

91. $V_5 = 155{,}000(0.70)^5 = \$26{,}050.85$

93. $A = \sum_{n=1}^{60} 100\left(1 + \frac{0.06}{12}\right)^n = 100\left(1 + \frac{0.06}{12}\right) \cdot \frac{\left[1 - \left(1 + \frac{0.06}{12}\right)^{60}\right]}{\left[1 - \left(1 + \frac{0.06}{12}\right)\right]} \approx \$7011.89$

95. Let $N = 12t$ be the total number of deposits.

$$\begin{aligned}
A &= P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^N \\
&= \left(1 + \frac{r}{12}\right)\left[P + P\left(r + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{N-1}\right] \\
&= P\left(1 + \frac{r}{12}\right)\sum_{n=1}^{N}\left(1 + \frac{r}{12}\right)^{n-1} \\
&= P\left(1 + \frac{r}{12}\right)\frac{1 - \left(1 + \frac{r}{12}\right)^N}{1 - \left(1 + \frac{r}{12}\right)} \\
&= P\left(1 + \frac{r}{12}\right)\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^N\right] \\
&= P\left(\frac{12}{r} + 1\right)\left[-1 + \left(1 + \frac{r}{12}\right)^N\right] \\
&= P\left[\left(1 + \frac{r}{12}\right)^N - 1\right]\left(1 + \frac{12}{r}\right) \\
&= P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right)
\end{aligned}$$

97. $P = \$50,\ r = 7\%,\ t = 20$ years

(a) Compounded monthly: $A = 50\left[\left(1 + \frac{0.07}{12}\right)^{12(20)} - 1\right]\left(1 + \frac{12}{0.07}\right) \approx \$26{,}198.27$

(b) Compounded continuously: $A = \frac{50e^{0.07/12}(e^{0.07(20)} - 1)}{e^{0.07/12} - 1} \approx \$26{,}263.88$

99. $P = \$100,\ r = 10\%,\ t = 40$ years

(a) Compounded monthly: $A = 100\left[\left(1 + \frac{0.10}{12}\right)^{12(40)} - 1\right]\left(1 + \frac{12}{0.10}\right) \approx \$637{,}678.02$

(b) Compounded continuously: $A = \frac{100e^{0.10/12}(e^{(0.10)(40)} - 1)}{e^{0.10/12} - 1} \approx \$645{,}861.43$

101. $P = W\sum_{n=1}^{12t}\left[\left(1 + \frac{r}{12}\right)^{-1}\right]^n$

$$= W\left(1 + \frac{r}{12}\right)^{-1}\left[\frac{1 - \left(1 + \frac{r}{12}\right)^{-12t}}{1 - \left(1 - \frac{r}{12}\right)^{-1}}\right]$$

$$= W\left(\frac{1}{1 + \frac{r}{12}}\right)\frac{\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]}{1 - \frac{1}{\left(1 + \frac{r}{12}\right)}}$$

$$= W\frac{\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]}{\left(1 + \frac{r}{12}\right) - 1}$$

$$= W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]$$

103. $\sum_{n=0}^{5}\frac{16^2}{4}\left(\frac{1}{2}\right)^n \approx 126$

Total area of shaded region is approximately 126 square inches.

105. $\sum_{n=0}^{6} 3.978e^{0.11n}$, $a_1 = 3.978$, $r = e^{0.11}$

Sum $= a_1\left(\frac{1 - r^n}{1 - r}\right) = 3.978\,\frac{1 - e^{0.11(7)}}{1 - e^{0.11}} \approx 39.68$ billion

107. $S_n = \sum_{i=1}^{n} 0.01(2)^{i-1}$

$S_{29} = \$5{,}368{,}709.11$

$S_{30} = \$10{,}737{,}418.23$

$S_{31} = \$21{,}474{,}836.47$

109. False. See definition page 638.

111. True. The sequence is $a, a, a, \ldots$ which is arithmetic $(d = 0)$.

113. $a_1 = 8$

$a_2 = 8\left(\frac{2x}{3}\right) = \frac{16}{3}x$

$a_3 = \frac{16}{3}x\left(\frac{2x}{3}\right) = \frac{32x^2}{9} = \frac{2^5x^2}{3^2}$

$a_4 = \frac{2^5x^2}{3^2}\left(\frac{2x}{3}\right) = \frac{2^6x^3}{3^3}$

$a_5 = \frac{2^7x^4}{3^4}$

115. $a_1 = \frac{1}{2}$

$a_2 = \frac{1}{2}(7x) = \frac{7x}{2}$

$a_3 = \frac{7x}{2}(7x) = \frac{7^2x^2}{2}$

$a_4 = \frac{7^3x^3}{2}$

$a_5 = \frac{7^4x^4}{2}$

117. $a_1 = 6, r = 3e^x, n = 8$

$a_n = a_1r^{n-1}$

$a_8 = 6(3e^x)^7 = 13{,}122e^{7x}$

119. $a_1 = 4, r = \frac{4x}{3}, n = 6$

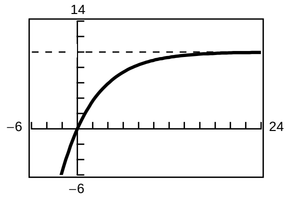
$a_n = a_1r^{n-1}$

$a_6 = 4\left(\frac{4x}{3}\right)^5 = \frac{4096}{243}x^5$

121. $f(x) = 2\left[\frac{1 - 0.8^x}{1 - 0.8}\right]$

$$\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n = \frac{2}{1 - \frac{4}{5}} = 10$$

The horizontal asymptote of $f(x)$ is $y = 10$. This corresponds to the sum of the series.



123. To use the first two terms of a geometric series to find the n^{th} term, first divide the second term by the first term to obtain the constant ratio. The n^{th} term is the first term multiplied by the common ratio raised to the $(n - 1)$ power.

$r = \frac{a_2}{a_1}, a_n = a_1r^{n-1}$

125. $\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{200}{50} + \frac{200}{42} = 200\left[\frac{92}{2100}\right]$ hours

$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{400}{200\left[\frac{92}{2100}\right]} = \frac{2(2100)}{92} \approx 45.65$ mph

127. Your friend mows at the rate of $\frac{1}{4}$ lawns/hour, and your rate is $\frac{1}{6}$ lawns/hour.Together, the time would be

$\frac{1}{\frac{1}{4} + \frac{1}{6}} = \frac{1}{\frac{10}{24}} = \frac{24}{10} = 2.4$ hours.

129. $-4\begin{bmatrix} 7 & 2 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} -5 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -33 & -3 \\ 27 & 1 \end{bmatrix}$

131. $\begin{bmatrix} 2 & 6 & -8 \\ 12 & 4 & -20 \\ 4 & 2 & 10 \end{bmatrix}$

133. $\sum_{i=0}^{6} 4i^2 = 4[0 + 1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2] = 364$

135. $\sum_{k=0}^{4} \frac{2}{k^2 + 2} = \frac{2}{2} + \frac{2}{3} + \frac{2}{6} + \frac{2}{11} + \frac{2}{18} = \frac{227}{99} \approx 2.293$