Section 9.4 Mathematical Induction

- You should be sure that you understand the principle of mathematical induction. If P_n is a statement involving the positive integer n, where P_1 is true and the truth of P_k implies the truth of P_{k+1}, then P_n is true for all positive integers n.
- You should be able to verify (by induction) the formulas for the sums of powers of integers and be able to use these formulas.
- You should be able to work with finite differences.

Solutions to Odd-Numbered Exercises

1. $P_k = \dfrac{5}{k(k+1)}$

$$P_{k+1} = \frac{5}{(k+1)[(k+1)+1]} = \frac{5}{(k+1)(k+2)}$$

3. $P_k = \dfrac{k^2(k+3)^2}{6}$

$$P_{k+1} = \frac{(k+1)^2[(k+1)+3]^2}{6} = \frac{(k+1)^2(k+4)^2}{6}$$

5. $P_k = 1 + 6 + 11 + \cdots + [5(k-1) - 4] + [5k - 4]$

$$P_{k+1} = 1 + 6 + 11 + \cdots + [5k - 4] + [5(k+1) - 4]$$
$$= 1 + 6 + 11 + \cdots + [5k - 4] + [5k + 1]$$

7. 1. When $n = 1$, $S_1 = 2 = 1(1 + 1)$.

2. Assume that

$$S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k+1).$$

Then,

$$S_{k+1} = 2 + 4 + 6 + 8 + \cdots + 2k + 2(k+1)$$
$$= S_k + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2).$$

We conclude by mathematical induction that the formula is valid for all positive integer values of n.

9. 1. When $n = 1$, $S_1 = 3 = \frac{1}{2}(5(1) + 1)$

2. Assume that $S_k = 3 + 8 + 13 + \cdots + (5k - 2) = \frac{k}{2}(5k + 1)$

Then: $S_{k+1} = 3 + 8 + 13 + \cdots + (5k - 2) + [5(k+1) - 2]$

$$= S_k + [5k + 3] = \frac{k}{2}(5k + 1) + 5k + 3$$
$$= \frac{1}{2}[5k^2 + 11k + 6] = \frac{1}{2}(k+1)(5k+6)$$
$$= \frac{1}{2}(k+1)(5(k+1) + 1)$$

We conclude by mathematical induction that the formula is valid for all positive integers n.

11. 1. When $n = 1$, $S_1 = 1 = 2^1 - 1$.

2. Assume that

$$S_k = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1.$$

Then,

$$S_{k+1} = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^k$$
$$= S_k + 2^k = 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1.$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

13. 1. When $n = 1$, $S_1 = 1 = \frac{1(1 + 1)}{2}$.

2. Assume that

$$S_k = 1 + 2 + 3 + 4 + \cdots + k = \frac{k(k + 1)}{2}.$$

Then,

$$S_{k+1} = 1 + 2 + 3 + 4 + \cdots + k + (k + 1)$$
$$= S_k + (k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}.$$

Therefore, we conclude that this formula holds for all positive integer values of n.

15. 1. When $n = 1$, $S_1 = 1^2 = \frac{1(2(1) - 1)(2(1) + 1)}{3}$

2. Assume that $S = 1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$

Then, $S_{k+1} = 1^2 + 3^2 + \cdots + (2k - 1)^2 + (2k + 1)^2$

$$= S_k + (2k + 1)^2 = \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2$$
$$= (2k + 1)\left[\frac{k(2k - 1)}{3} + (2k + 1)\right] = \frac{2k + 1}{3}[2k^2 - k + 6k + 3]$$
$$= \frac{2k + 1}{3}(2k + 3)(k + 1) = \frac{(k + 1)(2(k + 1) - 1)(2(k + 1) + 1)}{3}$$

Therefore, we conclude that the formula is valid for all positive integers n.

17. 1. When $n = 1$,

$$S_1 = 1^4 = \frac{1(1 + 1)(2 \cdot 1 + 1)(3 \cdot 1^2 + 3 \cdot 1 - 1)}{30}.$$

2. Assume that $S_k = \sum_{i=1}^{k} i^4 = \frac{k(k + 1)(2k + 1)(3k^2 + 3k - 1)}{30}$.

Then, $S_{k+1} = S_k + (k + 1)^4$

$$= \frac{k(k + 1)(2k + 1)(3k^2 + 3k - 1)}{30} + (k + 1)^4 = \frac{k(k + 1)(2k + 1)(3k^2 + 3k - 1) + 30(k + 1)^4}{30}$$

$$= \frac{(k + 1)[k(2k + 1)(3k^2 + 3k - 1) + 30(k + 1)^3]}{30} = \frac{(k + 1)(6k^4 + 39k^3 + 91k^2 + 89k + 30)}{30}$$

$$= \frac{(k + 1)(k + 2)(2k + 3)(3k^2 + 9k + 5)}{30} = \frac{(k + 1)(k + 2)(2(k + 1) + 1)(3(k + 1)^2 + 3(k + 1) -}{30}$$

Therefore, we conclude that this formula holds for all positive integer values of n.

19. 1. When $n = 1$, $S_1 = 2 = \frac{1(2)(3)}{3}$.

2. Assume that

$$S_k = 1(2) + 2(3) + 3(4) + \cdots + k(k + 1) = \frac{k(k + 1)(k + 2)}{3}.$$

Then,

$$S_{k+1} = 1(2) + 2(3) + 3(4) + \cdots + k(k + 1) + (k + 1)(k + 2)$$

$$= S_k + (k + 1)(k + 2) = \frac{k(k + 1)(k + 2)}{3} + \frac{3(k + 1)(k + 2)}{3}$$

$$= \frac{(k + 1)(k + 2)(k + 3)}{3}.$$

Thus, this formula is valid for all positive integer values of n.

21. 1. When $n = 4$, $4! = 24$ and $2^4 = 16$, thus $4! > 2^4$.

2. Assume $k! > 2^k$, $k > 4$. Then, $(k + 1)! = k!(k + 1) > 2^k(2)$ since $k + 1 > 2$. Thus, $(k + 1)! > 2^{k+1}$.

Therefore, by mathematical induction, the formula is valid for all integers n such that $n \geq 4$.

23. 1. When $n = 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \approx 1.707$ and $\sqrt{2} \approx 1.414$, thus $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$.

2. Assume

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k},\ k > 2.$$

Then,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Now we need to show that

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1},\ \ k > 2.$$

This is true because

$$\sqrt{k(k+1)} > k$$
$$\sqrt{k(k+1)} + 1 > k + 1$$
$$\frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} > \frac{k+1}{\sqrt{k+1}}$$
$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

Therefore,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

Therefore, by mathematical induction, the formula is valid for all integers n such that $n \geq 2$.

25. 1. When $n = 1$, $1 + a \geq a$ since $1 > 0$.

2. Assume $(1 + a)^k \geq ka$

Then

$$\begin{aligned} (1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq ka(1+a) \\ &= ka + ka^2 \\ &\geq ka + k \qquad \text{(because } a > 1) \\ &= (k+1)a \end{aligned}$$

Therefore, by mathematical induction, the inequality is valid for all integers $n \geq 1$.

27. 1. When $n = 1$, $(ab)^1 = a^1b^1 = ab$.

2. Assume that $(ab)^k = a^kb^k$.

Then,

$$\begin{aligned} (ab)^{k+1} &= (ab)^k(ab) \\ &= a^kb^kab \\ &= a^{k+1}b^{k+1}. \end{aligned}$$

Thus, $(ab)^n = a^nb^n$.

29. 1. When $n = 1$, $(x_1)^{-1} = x_1^{-1}$.

2. Assume that

$$(x_1x_2x_3 \cdots x_k)^{-1} = x_1^{-1}x_2^{-1}x_3^{-1} \cdots x_k^{-1}.$$

Then,

$$\begin{aligned} (x_1x_2x_3 \cdots x_kx_{k+1})^{-1} &= [(x_1x_2x_3 \cdots x_k)x_{k+1}]^{-1} \\ &= (x_1x_2x_3 \ldots x_k)^{-1}x_{k+1}^{-1} \\ &= x_1^{-1}x_2^{-1}x_3^{-1} \cdots x_k^{-1}x_{k+1}^{-1}. \end{aligned}$$

Thus, the formula is valid.

31. 1. When $n = 1$, $x(y_1) = xy_1$.

2. Assume that

$$x(y_1 + y_2 + \cdots + y_k) = xy_1 + xy_2 + \cdots + xy_k.$$

Then,

$$\begin{aligned} xy_1 + xy_2 + \cdots + xy_k + xy_{k+1} &= x(y_1 + y_2 + \cdots + y_k) + xy_{k+1} \\ &= x[(y_1 + y_2 + \cdots + y_k) + y_{k+1}] \\ &= x(y_1 + y_2 + \cdots + y_k + y_{k+1}). \end{aligned}$$

Hence, the formula holds.

33. 1. When $n = 1$, $[1^3 + 3(1)^2 + 2(1)] = 6$ and 3 is a factor.

2. Assume that 3 is a factor of $(k^3 + 3k^2 + 2k)$. Then,

$$\begin{aligned} [(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)] &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\ &= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 6) \\ &= (k^3 + 3k^2 + 2k) + 3(k^2 + 3k + 2). \end{aligned}$$

Since 3 is a factor of $(k^3 + 3k^2 + 2k)$ by our assumption, and 3 it is a factor of $3(k^2 + 3k + 2)$ then 3 is a factor of the whole sum.

Thus, 3 is a factor of $(n^3 + 3n^2 + 2n)$ for every positive integer n.

35. 1. When $n = 2$, $[9^2 - 8(2) - 1] = 64$ and 64 is a factor.

2. Assume that 64 is a factor of $(9^k - 8k - 1)$. Then $9^{k+1} - 8(k + 1) - 1 = 9^k \cdot 9 - 8k - 9$
$= 9[9^k - 1 - 8k] + 64k$ Since 64 is a factor of $(9^k - 8k - 1)$ is a factor of $9(9^k - 8k - 1) + 64k$.

Therefore, by mathematical induction, the statement is true for all integers $n \geq 2$.

37. $a_0 = 10, \quad a_n = 4a_{n-1}$

$a_0 = 10$

$a_1 = 4(10) = 40$

$a_2 = 4(40) = 160$

$a_3 = 4(160) = 640$

$a_4 = 4(640) = 2560$

39. $a_0 = 0, \quad a_1 = 2, \quad a_n = a_{n-1} + 2a_{n-2}$

$a_0 = 0$

$a_1 = 2$

$a_2 = 2 + 2(0) = 2$

$a_3 = 2 + 2(2) = 6$

$a_4 = 6 + 2(2) = 10$

41. $a_1 = 2, \quad a_n = n - a_{n-1}$

$a_1 = 2$

$a_2 = n - a_1 = 2 - 2 = 0$

$a_3 = n - a_2 = 3 - 0 = 3$

$a_4 = n - a_3 = 4 - 3 = 1$

$a_5 = n - a_5 = 5 - 1 = 4$

a_n: 2 0 3 1 4

First differences: -2 3 -2 3

Second differences: 5 -5 5

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

43. $a_2 = -3, \quad a_n = -2a_{n-1}$

$a_2 = -3$

$a_3 = -2a_2 = -2(-3) = 6$

$a_4 = -2a_3 = -2(6) = -12$

$a_5 = -2a_4 = -2(-12) = 24$

$a_6 = -2a_5 = -2(24) = -48$

a_n: -3 6 -12 24 -48

First differences: 9 -18 36 -72

Second differences: -27 54 -108

Since neither the first nor the second differences are equal, the sequence does not have a linear or quadratic model.

45. $a_0 = 2, \quad a_n = (a_{n-1})^2$

$a_0 = 2$

$a_1 = a_0{}^2 = 2^2 = 4$

$a_2 = a_1{}^2 = 4^2 = 16$

$a_3 = a_2{}^2 = 16^2 = 256$

$a_4 = a_3{}^2 = 256^2 = 65{,}536$

a_n: 2, 4, 16, 256, 65,536

First differences: 2, 12, 240, 65,280

Second differences: 10, 228, 65,040

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

47. $a_1 = 0, \quad a_n = a_{n-1} + 2n$

$a_1 = 0$

$a_2 = a_1 + 2(2) = 0 + 4 = 4$

$a_3 = a_2 + 2(3) = 4 + 6 = 10$

$a_4 = a_3 + 2(4) = 10 + 8 = 18$

$a_5 = a_4 + 2(5) = 18 + 10 = 28$

a_n: 0, 4, 10, 18, 28

First differences: 4, 6, 8, 10

Second differences: 2, 2, 2

Since the second differences are equal, the sequence has a quadratic model.

49. $a_0 = 0, \quad a_n = a_{n-1} - 1$

$a_0 = 0$

$a_1 = a_0 - 1 = 0 - 1 = -1$

$a_2 = a_1 - 1 = -1 - 1 = -2$

$a_3 = a_2 - 1 = -2 - 1 = -3$

$a_4 = a_3 - 1 = -3 - 1 = -4$

a_n: 0, −1, −2, −3, −4

First differences: −1, −1, −1, −1

Second differences: 0, 0, 0

Since the first differences are equal, the sequence has a linear model.

51. $a_0 = 7, \quad a_1 = 6, \quad a_3 = 10$

Let $a_n = an^2 + bn + c$. Thus,

$a_0 = a(0)^2 + b(0) + c = 7 \implies c = 7$

$a_1 = a(1)^2 + b(1) + c = 6 \implies a + b + c = 6$

$a + b = -1$

$a_3 = a(3)^2 + b(3) + c = 10 \implies 9a + 3b + c = 10$

$9a + 3b = 3$

$3a + b = 1$

By elimination: $-a - b = 1$

$3a + b = 1$

$2a = 2$

$a = 1 \implies b = -2$

Thus, $a_n = n^2 - 2n + 7$.

53. $a_0 = 3, \quad a_2 = 0, \quad a_6 = 36$

Let $a_n = an^2 + bn + c$. Thus,

$$\begin{aligned}
a_0 &= a(0)^2 + b(0) + c = 3 \Longrightarrow & c &= 3\\
a_2 &= a(2)^2 + b(2) + c = 0 \Longrightarrow & 4a + 2b + c &= 0\\
& & 4a + 2b &= -3\\
a_6 &= a(6)^2 + b(6) + c = 36 \Longrightarrow & 36a + 6b + c &= 36\\
& & 36a + 6b &= 33\\
& & 12a + 2b &= 11
\end{aligned}$$

By elimination:

$$\begin{aligned}
-4a - 2b &= 3\\
12a + 2b &= 11\\
8a &= 14\\
a &= \tfrac{7}{4} \Longrightarrow b = -5
\end{aligned}$$

Thus, $a_n = \frac{7}{4}n^2 - 5n + 3$.

55. False. See page 653.

57. See the domino illustration and Figure 9.11.

59. $\begin{bmatrix} 1 & -1 & : & 2 \\ -4 & 5 & : & -3 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & : & 7 \\ 0 & 1 & : & 5 \end{bmatrix}$

Answers: (7, 5)

61.

$$\begin{aligned}
y &= x^2\\
-3x + 2y = 2 \quad &\Longrightarrow \quad -3x + 2x^2 = 2\\
& 2x^2 - 3x - 2 = 0\\
& (2x + 1)(x - 2) = 0\\
& x = -\tfrac{1}{2} \quad \text{or } x = 2
\end{aligned}$$

$x = -\frac{1}{2} \Longrightarrow y = \frac{1}{4}, x = 2 \Longrightarrow y = 4$

Points of intersection: $\left(-\frac{1}{2}, \frac{1}{4}\right)$, (2, 4)

63.

$$\begin{aligned}
x - y \phantom{{}+2z} &= -1\\
x + 2y - 2z &= 3\\
3x - y + 2z &= 3
\end{aligned}$$

Using an augmented matrix, we have

$$\begin{bmatrix} 1 & -1 & 0 & \vdots & -1 \\ 1 & 2 & -2 & \vdots & 3 \\ 3 & -1 & 2 & \vdots & 3 \end{bmatrix}$$

$$\begin{matrix} \\ -R_1 + R_2 \rightarrow \\ -3R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 0 & \vdots & -1 \\ 0 & 3 & -2 & \vdots & 4 \\ 0 & 2 & 2 & \vdots & 6 \end{bmatrix}$$

$$\begin{matrix} \\ -R_3 + R_2 \rightarrow \\ \frac{1}{2}R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 0 & \vdots & -1 \\ 0 & 1 & -4 & \vdots & -2 \\ 0 & 1 & 1 & \vdots & 3 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \rightarrow \\ \\ -R_2 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -4 & \vdots & -3 \\ 0 & 1 & -4 & \vdots & -2 \\ 0 & 0 & 5 & \vdots & 5 \end{bmatrix}$$

$$\begin{matrix} 4R_3 + R_1 \rightarrow \\ 4R_3 + R_2 \rightarrow \\ \frac{1}{5}R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

Thus, $x = 1, y = 2, z = 1$.

Answer: (1, 2, 1)