Section 9.6 Counting Principles

- You should know The Fundamental Counting Principle.
- ${}_nP_r = \dfrac{n!}{(n-r)!}$ is the number of permutations of n elements taken r at a time.
- Given a set of n objects that has n_1 of one kind, n_2 of a second kind, and so on, the number of distinguishable permutations is

  $$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

- ${}_nC_r = \dfrac{n!}{(n-r)!r!}$ is the number of combinations of n elements taken r at a time.

Solutions to Odd-Numbered Exercises

1. Odd integers: 1, 3, 5, 7, 9, 11

6 ways

3. Prime integers: 2, 3, 5, 7, 11

5 ways

5. Divisible by 4: 4, 8, 12

3 ways

7. Sum is 10: $1+9, 2+8, 3+7, 4+6, 5+5, 6+4, 7+3, 8+2, 9+1$

9 ways

9. Amplifiers: 4 choices

Compact disc players: 6 choices

Speakers: 8 choices

Total: $4 \cdot 6 \cdot 8 = 192$ ways

11. Chemist: 3 choices

Statistician: 6 choices

Total: $3 \cdot 6 = 18$ ways

13. $10! = 3{,}628{,}800$ ways

15. 1^{st} Position: 2 choices

2^{nd} Position: 3 choices

3^{rd} Position: 2 choices

4^{th} Position: 1 choice

Total: $2 \cdot 3 \cdot 2 \cdot 1 = 12$ ways

Label the four people A, B, C, and D and suppose that A and B are willing to take the first position. The twelve combinations are as follows.

ABCD	BACD
ABDC	BADC
ACBD	BCAD
ACDB	BCDA
ADBC	BDAC
ADCB	BDCA

17. $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}760{,}000$

19. (a) $9 \cdot 10 \cdot 10 = 900$

(b) $9 \cdot 9 \cdot 8 = 648$

(c) $9 \cdot 10 \cdot 2 = 180$

(d) $10 \cdot 10 \cdot 10 - 400 = 600$

21. $40^3 = 64{,}000$

23. (a) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

(b) $6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 48$

25. ${}_nP_r = \dfrac{n!}{(n-r)!}$

So, ${}_4P_4 = \dfrac{4!}{0!} = 4! = 24.$

27. ${}_8P_3 = \dfrac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$

29. ${}_5P_4 = \dfrac{5!}{1!} = 120$

31. $14 \cdot {}_nP_3 = {}_{n+2}P_4$ Note $n \geq 3$ for this to be defined.

$$14\left[\frac{n!}{(n-3)!}\right] = \frac{(n+2)!}{(n-2)!}$$

$14n(n-1)(n-2) = (n+2)(n+1)n(n-1)$ (We can divide here by $n(n-1)$ since $n \neq 0$, $n \neq 1$.)

$$14n - 28 = n^2 + 3n + 2$$

$$0 = n^2 - 11n + 30$$

$$0 = (n-5)(n-6)$$

$$n = 5 \quad \text{or} \quad n = 6$$

33. ${}_{20}P_6 = 27{,}907{,}200$

35. ${}_{120}P_4 = 197{,}149{,}680$

37. ${}_{20}C_4 = 4845$

39. $5! = 120$ ways

41. (a) $2^4 = 16$ characters

(b) $2 + 2^2 + 2^3 = 14$ characters

43. $\dfrac{7!}{2!1!3!1!} = \dfrac{7!}{2!3!} = 420$

45. $\dfrac{7!}{2!1!1!1!1!1!} = \dfrac{7!}{2!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

47.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

49. ${}_6C_2 = 15$

The 15 ways are listed below.

AB, AC, AD, AE, AF,
BC, BD, BE, BF, CD,
CE, CF, DE, DF, EF

51. ${}_{20}C_4 = 4845$ groups

53. ${}_{40}C_6 = 3{,}838{,}380$ ways

55. ${}_{100}C_5 = 75{,}287{,}520$ subsets

57. ${}_9C_2 = 36$ lines

59. (a) ${}_{12}C_4 = 495$ ways

(b) $({}_5C_2)({}_7C_2) = (10)(21) = 210$ ways

61. (a) ${}_8C_4 = \dfrac{8!}{4!4!} = 70$ ways

(b) There are $2^4 = 16$ ways that a group of four can be formed without any couples in the group. Therefore, if at least one couple is to be in the group, there are $70 - 16 = 54$ ways that could occur.

(c) $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways

63. ${}_5C_2 - 5 = 10 - 5 = 5$ diagonals

65. ${}_8C_2 - 8 = 28 - 8 = 20$ diagonals

67. False. Order matters in a permutation.

69. False. for example, ${}_1P_1 = {}_1C_1 = 1$

71. The symbol ${}_nP_r$ means the number of ways to choose and order r elements out of a set of n elements.

73. (b) ${}_{10}P_6$ is larger than ${}_{10}C_6$ because the permutations count different orderings as distinct.

75. $${}_nC_n = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = \frac{n!}{n!0!} = \frac{n!}{(n-0)!0!} = {}_nC_0$$

77. $$\begin{aligned} {}_nC_r &= \frac{n!}{(n-r)!r!} \\ &= \frac{1}{r!}\left[\frac{n!}{(n-r)^1}\right] \\ &= \frac{{}_nP_r}{r!} \end{aligned}$$

79. $$\begin{aligned} \frac{4}{t} + \frac{3}{2t} &= 1 \\ \frac{8+3}{2t} &= 1 \\ 11 &= 2t \\ t &= \frac{11}{2} = 5.5 \end{aligned}$$

81. $$\begin{aligned} e^{x/3} &= 16 \\ \frac{x}{3} &= \ln 16 \\ x &= 3 \ln 16 \approx 8.32 \end{aligned}$$

83. $$x = \frac{\begin{vmatrix} 35 & 1 \\ 10 & 2 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}} = \frac{60}{10} = 6$$

$$y = \frac{\begin{vmatrix} 8 & 35 \\ 6 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}} = \frac{-130}{10} = -13$$

Answer: $(6, -13)$

85. $$x = \frac{\begin{vmatrix} -74 & -11 \\ 8 & -4 \end{vmatrix}}{\begin{vmatrix} 10 & -11 \\ -8 & -4 \end{vmatrix}} = \frac{384}{-128} = -3$$

$$y = \frac{\begin{vmatrix} 10 & -74 \\ -8 & 8 \end{vmatrix}}{\begin{vmatrix} 10 & -11 \\ -8 & -4 \end{vmatrix}} = \frac{-512}{-128} = 4$$

Answer: $(-3, 4)$

87. $(x-1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

89. $(3x - y)^4 = 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$