CHAPTER 5
Analytic Trigonometry

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Section 5.1 Using Fundamental Identities

- You should know the fundamental trigonometric identities.

  (a) Reciprocal Identities

  $$\sin u = \frac{1}{\csc u} \qquad \csc u = \frac{1}{\sin u}$$

  $$\cos u = \frac{1}{\sec u} \qquad \sec u = \frac{1}{\cos u}$$

  $$\tan u = \frac{1}{\cot u} = \frac{\sin u}{\cos u} \qquad \cot u = \frac{1}{\tan u} = \frac{\cos u}{\sin u}$$

  (b) Pythagorean Identities

  $$\sin^2 u + \cos^2 u = 1$$

  $$1 + \tan^2 u = \sec^2 u$$

  $$1 + \cot^2 u = \csc^2 u$$

  (c) Cofunction Identities

  $$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

  $$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

  $$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

  (d) Negative Angle Identities

  $$\sin(-x) = -\sin x \qquad \csc(-x) = -\csc x$$

  $$\cos(-x) = \cos x \qquad \sec(-x) = \sec x$$

  $$\tan(-x) = -\tan x \qquad \cot(-x) = -\cot x$$

- You should be able to use these fundamental identities to find function values.
- You should be able to convert trigonometric expressions to equivalent forms by using the fundamental identities.
- You should be able to check your answers with a graphing utility.

Solutions to Odd-Numbered Exercises

1. $\sin x = \frac{\sqrt{3}}{2}, \cos x = \frac{1}{2} \implies x$ is in Quadrant I

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec x = \frac{1}{\cos x} = 2$$

$$\csc x = \frac{1}{\sin x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

3. $\sec \theta = \sqrt{2},\ \sin\theta = -\frac{\sqrt{2}}{2} \implies \theta$ is in Quadrant IV.

$$\cos \theta = \frac{1}{\sec \theta} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$$

$$\cot \theta = \frac{1}{\tan \theta} = -1$$

$$\csc \theta = -\sqrt{2}$$

5. $\tan x = \frac{7}{24}, \sec x = \frac{-25}{24} \implies x$ is in Quadrant III

$$\cot x = \frac{24}{7}$$

$$\cos x = -\frac{24}{25}$$

$$\sin x = -\sqrt{1 - \cos^2 x} = -\frac{7}{25}$$

$$\csc x = \frac{1}{\sin x} = -\frac{25}{7}$$

7. $\sec \phi = -1,\ \sin \phi = 0 \implies \phi = \pi$

$\cos \phi = -1$

$\tan \phi = 0$

$\cot \phi$ is undefined.

$\csc \phi$ is undefined.

9. $\sin(-x) = -\sin x = -\frac{2}{3} \implies \sin x = \frac{2}{3}$

$\sin x = \frac{2}{3},\ \tan x = -\frac{2\sqrt{5}}{5} \implies x$ is in Quadrant II.

$$\cos x = -\sqrt{1 - \sin^2 x} = -\sqrt{1 - \frac{4}{9}} = -\frac{\sqrt{5}}{3}$$

$$\cot x = \frac{1}{\tan x} = -\frac{\sqrt{5}}{2}$$

$$\sec x = \frac{1}{\cos x} = -\frac{3\sqrt{5}}{5}$$

$$\csc x = \frac{1}{\sin x} = \frac{3}{2}$$

11. $\tan \theta = 4, \sin \theta < 0 \implies \theta$ is in Quadrant III

$$\sec \theta = -\sqrt{\tan^2 \theta + 1} = -\sqrt{17}$$

$$\cos \theta = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$$

$$\cot \theta = \frac{1}{4}$$

$$\sin \theta = -\sqrt{1 - \cos^2 \theta} = -\sqrt{1 - \frac{1}{17}} = -\frac{4}{\sqrt{17}}$$

$$= \frac{-4\sqrt{17}}{17}$$

$$\csc \theta = -\frac{\sqrt{17}}{4}$$

13. $\sin \theta = -1,\ \cot \theta = 0 \implies \theta = \frac{3\pi}{2}$

$\cos \theta = \sqrt{1 - \sin^2 \theta} = 0$

$\sec \theta$ is undefined.

$\tan \theta$ is undefined.

$\csc \theta = -1$

15. By looking at the basic graphs of $\sin x$ and $\csc x$, we see that as $x \longrightarrow \frac{\pi}{2}^-$, $\sin x \longrightarrow 1$ and $\csc x \longrightarrow 1$.

17. By looking at the basic graphs of tan x and cot x, we see that as $x \longrightarrow \frac{\pi}{2}^-$, $\tan x \longrightarrow \infty$ and $\cot x \longrightarrow 0$.

19. $\csc x \sin x = \frac{1}{\sin x} \sin x = 1$. Matches (d)

21. $\tan^2 x - \sec^2 x = \tan^2 x - (\tan^2 x + 1) = -1$

The expression is matched with (a).

23. $\frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$

The expression is matched with (e).

25. $\cos x \csc x = \frac{\cos x}{\sin x} = \cot x$. Matches (b)

27. $\sec^4 x - \tan^4 x = (\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x)$

$= (\sec^2 x + \tan^2 x)(1) = \sec^2 x + \tan^2 x$

The expression is matched with (f).

29. $\frac{\sec^2 x - 1}{\sin^2 x} = \frac{\tan^2 x}{\sin^2 x} = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\sin^2 x} = \sec^2 x$

The expression is matched with (e).

31. $\cot x \sin x = \frac{\cos x}{\sin x} \sin x = \cos x$

33. $\sin \phi(\csc \phi - \sin \phi) = \sin \phi \csc \phi - \sin^2 \phi$

$= \sin \phi \cdot \frac{1}{\sin \phi} - \sin^2 \phi$

$= 1 - \sin^2 \phi$

$= \cos^2 \phi$

35. $\frac{\cot x}{\csc x} = \frac{\cos x/\sin x}{1/\sin x}$

$= \frac{\cos x}{\sin x} \cdot \frac{\sin x}{1} = \cos x$

37. $\sec \alpha \frac{\sin \alpha}{\tan \alpha} = \frac{1}{\cos \alpha}(\sin \alpha) \cot \alpha$

$= \frac{1}{\cos \alpha}(\sin \alpha)\left(\frac{\cos \alpha}{\sin \alpha}\right) = 1$

39. $\frac{\sin(-x)}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$

41. $\sin\left(\frac{\pi}{2} - x\right)\csc x = \cos x \cdot \frac{1}{\sin x} = \cot x$

43. $\frac{\cos^2 y}{1 - \sin y} = \frac{1 - \sin^2 y}{1 - \sin y}$

$= \frac{(1 + \sin y)(1 - \sin y)}{1 - \sin y}$

$= 1 + \sin y$

45. $\tan \phi \csc \phi = \frac{\sin \phi}{\cos \phi} \cdot \frac{1}{\sin \phi} = \frac{1}{\cos \phi} = \sec \phi$

47. $\frac{\csc \theta}{\sec \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \cot \theta + \cot \theta$

$= 2 \cot \theta$

49. $1 - \frac{\sin^2 \theta}{1 - \cos \theta} = \frac{1 - \cos \theta - \sin^2 \theta}{1 - \cos \theta} = \frac{\cos^2 \theta - \cos \theta}{1 - \cos \theta}$

$= \frac{\cos \theta(\cos \theta - 1)}{1 - \cos \theta} = -\cos \theta$

51. $\dfrac{\cot(-\theta)}{\csc\theta} = \dfrac{\cos(-\theta)}{\sin(-\theta)}\sin\theta = \dfrac{\cos\theta}{-\sin\theta}\sin\theta = -\cos\theta$

53. $$\begin{aligned}\sin\theta + \cos\theta\cot\theta &= \sin\theta + \cos\theta\frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta}\\ &= \csc\theta\end{aligned}$$

55. $$\begin{aligned}\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\\ &= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}\\ &= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\\ &= \frac{1+\sin\theta}{\cos\theta}\\ &= \sec\theta + \tan\theta\end{aligned}$$

57. $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = \sin^2\theta + \cos^2\theta = 1$

59. $$\begin{aligned}\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta} &= \frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta(1+\cos\theta)}\\ &= \frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)}\\ &= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\ &= \frac{2}{\sin\theta} = 2\csc\theta\end{aligned}$$

61. $\ln|\csc\theta| = \ln\left|\dfrac{1}{\sin\theta}\right| = \ln|\sin\theta|^{-1} = -\ln|\sin\theta|$

63. $\cot^2 x - \cot^2 x\cos^2 x = \cot^2 x(1-\cos^2 x) = \dfrac{\cos^2 x}{\sin^2 x}\sin^2 x = \cos^2 x$

65. $$\begin{aligned}\sin^2 x\sec^2 x - \sin^2 x &= \sin^2 x(\sec^2 x - 1)\\ &= \sin^2 x\tan^2 x\end{aligned}$$

67. $$\begin{aligned}\tan^4 x + 2\tan^2 x + 1 &= (\tan^2 x + 1)^2\\ &= (\sec^2 x)^2\\ &= \sec^4 x\end{aligned}$$

69. $$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)\\ &= (1)(\sin^2 x - \cos^2 x)\\ &= \sin^2 x - \cos^2 x\end{aligned}$$

71. $$\begin{aligned}(\sin x + \cos x)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x\\ &= (\sin^2 x + \cos^2 x) + 2\sin x\cos x\\ &= 1 + 2\sin x\cos x\end{aligned}$$

73. $(\sec x + 1)(\sec x - 1) = \sec^2 x - 1 = \tan^2 x$

75. $\dfrac{1}{1+\cos x}+\dfrac{1}{1-\cos x}=\dfrac{1-\cos x+1+\cos x}{(1+\cos x)(1-\cos x)}$

$=\dfrac{2}{1-\cos^2 x}$

$=\dfrac{2}{\sin^2 x}$

$=2\csc^2 x$

77. $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=\dfrac{\cos^2 x+(1+\sin x)^2}{\cos x(1+\sin x)}$

$=\dfrac{2+2\sin x}{\cos x(1+\sin x)}$

$=\dfrac{2(1+\sin x)}{\cos x(1+\sin x)}$

$=\dfrac{2}{\cos x}$

$=2\sec x$

79. $\dfrac{\sin^2 y}{1-\cos y}=\dfrac{1-\cos^2 y}{1-\cos y}$

$=\dfrac{(1+\cos y)(1-\cos y)}{1-\cos y}$

$=1+\cos y$

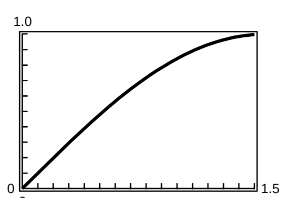
81. $\dfrac{3}{\sec x-\tan x}\cdot\dfrac{\sec x+\tan x}{\sec x+\tan x}=\dfrac{3(\sec x+\tan x)}{\sec^2 x-\tan^2 x}$

$=\dfrac{3(\sec x+\tan x)}{1}$

$=3(\sec x+\tan x)$

83. $y_1=\cos\left(\dfrac{\pi}{2}-x\right),\ y_2=\sin x$

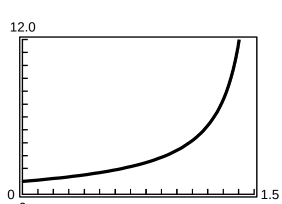
x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1	0.1987	0.3894	0.5646	0.7174	0.8415	0.9320	0.9854
y_2	0.1987	0.3894	0.5646	0.7174	0.8415	0.9320	0.9854



Conjecture: $y_1=y_2$

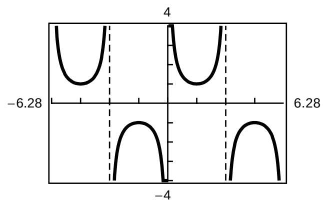
85. $y_1=\dfrac{\cos x}{1-\sin x},\ y_2=\dfrac{1+\sin x}{\cos x}$

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1	1.2230	1.5085	1.8958	2.4650	3.4082	5.3319	11.6814
y_2	1.2230	1.5085	1.8958	2.4650	3.4082	5.3319	11.6814

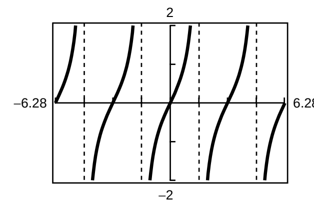


Conjecture: $y_1=y_2$

87. $y_1 = \cos x \cot x + \sin x = \csc x$



89. $y_1 = \sec x - \dfrac{\cos x}{1 + \sin x} = \tan x$



91. $\sqrt{25 - x^2} = \sqrt{25 - (5 \sin \theta)^2},\ x = 5 \sin \theta$

$$\begin{aligned} &= \sqrt{25 - 25 \sin^2 \theta} \\ &= \sqrt{25(1 - \sin^2 \theta)} \\ &= \sqrt{25 \cos^2 \theta} \\ &= 5 \cos \theta \end{aligned}$$

93. $\sqrt{x^2 - 9} = \sqrt{(3 \sec \theta)^2 - 9},\ x = 3 \sec \theta$

$$\begin{aligned} &= \sqrt{9 \sec^2 \theta - 9} \\ &= \sqrt{9 (\sec^2 \theta - 1)} \\ &= \sqrt{9 \tan^2 \theta} \\ &= 3 \tan \theta \end{aligned}$$

95. $\sqrt{x^2 + 25} = \sqrt{(5 \tan \theta)^2 + 25},\ x = 5 \tan \theta$

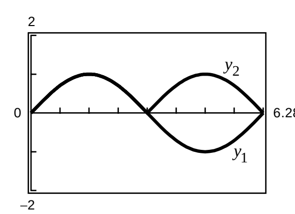
$$\begin{aligned} &= \sqrt{25 \tan^2 \theta + 25} \\ &= \sqrt{25(\tan^2 \theta + 1)} \\ &= \sqrt{25 \sec^2 \theta} \\ &= 5 \sec \theta \end{aligned}$$

97. $\sin \theta = \sqrt{1 - \cos^2 \theta}$

Let $y_1 = \sin x$ and $y_2 = \sqrt{1 - \cos^2 x},\ 0 \le x < 2\pi$.

$y_1 = y_2$ for $0 \le x \le \pi$, so we have

$\sin \theta = \sqrt{1 - \cos^2 \theta}$ for $0 \le \theta \le \pi$.

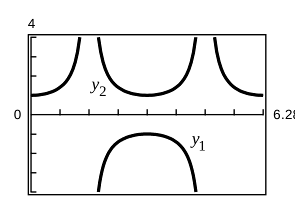


99. $\sec \theta = \sqrt{1 + \tan^2 \theta}$

Let $y_1 = \dfrac{1}{\cos x}$ and $y_2 = \sqrt{1 + \tan^2 x},\ 0 \le x < 2\pi$.

$y_1 = y_2$ for $0 \le x < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < x < 2\pi$, so we have

$\sin \theta = \sqrt{1 + \tan^2 \theta}$ for $0 \le \theta < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$.



101. $\ln|\cos \theta| - \ln|\sin \theta| = \ln \dfrac{|\cos \theta|}{|\sin \theta|} = \ln|\cot \theta|$

103. $\ln(1 + \sin x) - \ln|\sec x| = \ln\left|\dfrac{1 + \sin x}{\sec x}\right| = \ln|\cos x + \cos x \cdot \sin x|$

105. (a) $\csc^2 132^\circ - \cot^2 132^\circ \approx 1.8107 - 0.8107 = 1$

(b) $\csc^2 \dfrac{2\pi}{7} - \cot^2 \dfrac{2\pi}{7} \approx 1.6360 - 0.6360 = 1$

107. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

(a) $\theta = 80^{\circ}$

$\cos(90^{\circ} - 80^{\circ}) = \sin 80^{\circ}$

$0.9848 = 0.9848$

(b) $\theta = 0.8$

$\cos\left(\frac{\pi}{2} - 0.8\right) = \sin 0.8$

$0.7174 = 0.7174$

109. $\csc x \cot x - \cos x = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} - \cos x$

$= \cos x(\csc^2 x - 1)$

$= \cos x \cdot \cot^2 x$

111. False. $\frac{1}{5\cos\theta} = \frac{1}{5}\sec\theta$

113. False. $\sin\theta\csc\phi \neq 1$ unless $\theta = \phi$

115. $\cos\theta$

$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sqrt{1 - \cos^2\theta}}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta} = \pm\frac{1}{\sqrt{1 - \cos^2\theta}}$

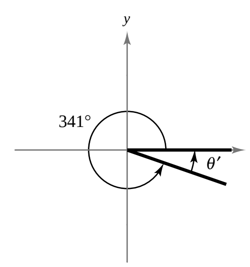
$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta} = \pm\frac{\cos\theta}{\sqrt{1 - \cos^2\theta}}$

The sign + or − depends on the choice of θ.

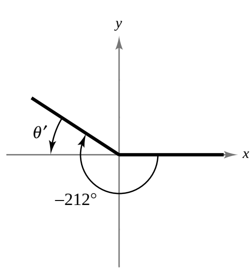
117. $\theta = 341^{\circ}$

$\theta' = 360^{\circ} - 341^{\circ} = 19^{\circ}$



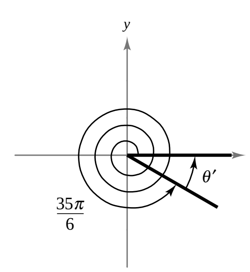
119. $\theta = -212^{\circ}$ is coterminal with 148°

$\theta' = 180^{\circ} - 148^{\circ} = 32^{\circ}$



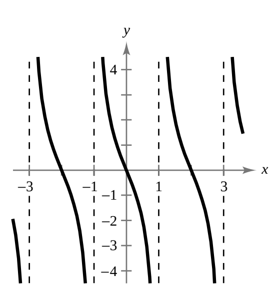
121. $\theta = \frac{35\pi}{6}$ is coterminal with $\frac{11\pi}{6}$

$\theta' = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$



123. $f(x) = -2\tan\frac{\pi x}{2}$

Period: $\frac{\pi}{\frac{\pi}{2}} = 2$



125. $f(x) = \frac{3}{2}\cos(x - \pi) + 3$

Amplitude: $\frac{3}{2}$